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JOÃO PAULO LIMA DE OLIVEIRA[1]

# Solution Curve For Control Systems On Lie Groups

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Tese apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática, Centro de Ciências Exatas, da Universidade Estadual de Maringá, como requisito parcial para obtenção do título de Doutor em Matemática.
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Orientador: Prof. Dr. Alexandre José Santana
Co-orientador: Prof. Dr. Simão Nicolau Stelmastchuk

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COMISSÃO JULGADORA:

Professor Dr. Alexandre José Santana - Universidade Estadual de Maringá (Presidente)
Professor Dr. Carlos Eduardo Durán Fernández – Universidade Federal do Paraná
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# Resumo

No contexto dos grupos de Lie, a Teoria de Controle se ocupa basicamente do estudo dos sistemas de controle invariantes, lineares, bilineares e afins. Para sistemas invariantes - considerando que as funções de controle são constantes por partes - as soluções do sistema têm uma descrição já bem conhecida (veja [24]). Isto nos leva ao primeiro objetivo deste trabalho: obter uma descrição explícita das trajetórias para os outros sistemas sob a hipótese de que os campos lineares comutam. A descrição destas trajetórias é obtida como a curva integral de um campo vetorial invariante conveniente em um produto semidireto de um grupo de Lie por um espaço euclidiano (como em [9]). Em particular, consideramos o caso em que as derivações associadas aos campos lineares são internas (o que ocorre, por exemplo, em toda álgebra de Lie semissimples). Neste caso, as soluções são descritas de uma maneira consideravelmente mais simples e elegante.

Deste ponto em diante, os resultados são aplicados à obtenção de novas proposições. Os resultados obtidos vão desde condições que relacionam a controlabilidade de sistemas de controle linear/afim com sistemas invariantes associados até o estudo de semiconjugação de sistemas por homomorfismos de grupos e propriedades de conjuntos estáveis.

**Palavras-chave:** sistema de controle linear, sistema de controle afim, soluções, controlabilidade, conjugação de sistemas, estabilidade.

# Abstract

In the context of Lie groups, Control Theory is primarily concerned with the study of invariant, linear, bilinear and affine control systems. For invariant systems - considering that the control functions are piecewise constant - the solutions of the system has a well known and good description (see [24]). This brings us to the first objective of this work: to give an explicit description of the solution curve for the other systems under the assumption that the linear vector fields commute. These solutions are obtained as the integral curve of a convenient invariant vector field on a semidirect product of a Lie group with an Euclidean space (just as in [9]). In particular, we consider the case where the derivations associated to the linear vector fields are inner (which occurs, for example, in every semi simple Lie algebra), in which case the solutions are described in a considerably simpler and more elegant way.

Thenceforth, our achievements are applied to obtain new propositions. The results range from expressions that relate the controllability of linear/affine control systems with associated invariant ones to the study of system semiconjugation by Lie group homomorphisms and properties of stability sets.

**Keywords:** linear control system, affine control system, solutions, controllability, system conjugation, stability.

# Contents

# Introduction

The main object of study in this thesis are control systems defined on connected Lie groups. We think it is reasonable to say that this class of systems has in *linear control system* one of its main representatives. Indeed, initially the study of control theory focused on this kind of system and currently a broad theory has been developed for them. A linear control system on $\mathbb{R}^n$ is given by a family of differential equations

$$\frac{dx}{dt} = Ax + Bu, \tag{1}$$

where $A \in M(n \times n; \mathbb{R})$, $B \in M(n \times m; \mathbb{R})$ and $u \in U \subset \mathbb{R}^m$ is a control parameter. One of the issues of interest in this field is to find among all the possible control functions one function $u$ in such a way that the system has specific properties, such as optimal or periodic trajectories, stabilization at a fixed point and reachability (controllability) from a determined initial position.

Over the years, lots of results concerning controllability aspects have been established. In the achievement of many of those results, the description of the solutions of the system played an important role (see for instance [1] and [10]). The solutions of this system are obtained through the formula of the variation of the constants that provides for each control function $u$ the solution of the corresponding differential equation (1). More specifically, given $u$ and an initial condition $x_0 \in \mathbb{R}^m$, the solution is written as

$$\phi(t, x_0, u) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds.$$

Currently, out of the context of euclidean spaces, linear control systems are studied in more general environments. They were considered initially by Markus in matrix Lie groups [19]. Later, Ayala and Tirao [5] extended the system for general Lie groups. In this broader context, a linear system on a connected Lie group $G$ is a control system of the form

$$\frac{dg}{dt} = \mathcal{X}(g) + \sum_{i=1}^{m} u_i(t)Y_i(g), \tag{2}$$

where $g \in G$, $\mathcal{X}$ is a linear vector field, $Y_1, \ldots, Y_m$[2] are invariant vector fields, and $u = (u_1, \ldots, u_m)$ is an admissible control parameter. Since then, this system has become the subject of study of several authors. Among them, we reference [4], [9], [12], [26] and so on.

Despite linear control systems (1) have a good description of their solutions, it is not true in the case of systems on general Lie groups (2). One of the first results in this way is found in [2]. The purpose of this work is to contribute in this direction. In principle, we intend to describe the solutions not only for linear control systems but also for *Bilinear* and, more generally, for *Affine Control Systems*. Thereafter, we intend to use this description of the system trajectories as a tool that allows us to get a clearer understanding of some important aspects related to control theory such as controllability, systems conjugation, dynamical behavior, etc.

In order to accomplish the task described above, this work is organized into four chapters as follows:

Chapter 1 consists of two sections: the first contains a brief exposition of the notions and background information of control theory in differentiable manifolds. We intend to be objective at this point so that only some basic concepts will be presented. For a more detailed exposition of the subject, we suggest the study of [1], [10] and [14]. The same can be said about Section 1.2. But this time, the objective is to introduce the notations and results concerning linear vector fields on Lie groups, a concept necessary for the definition of the control systems that will be studied next.

In Chapter 2 we begin to approach the problem of trying to describe the solutions of control system (2). Our idea for constructing the trajectories is to use a technique[3] presented in [9] that considers an invariant system on a semidirect product $G \times_\varphi \mathbb{R}$, where $\varphi$ is a one-parameter subgroup of automorphisms associated to the linear vector field of the system. Thenceforth, the solution curve is obtained as the integral curve of a certain invariant vector field (see Theorem (2.7)).

Reading the third chapter makes it clear that it is motivated by the previous one.

[2]Right or left invariant, depending on the approach.

[3]Cardetti and Mittenhuber [9] in this occasion use a semidirect product $G \times_\varphi \mathbb{R}$ to prove the well known *ad-rank condition* for local controllability.

Having described the solutions of linear systems, we now consider with a similar approach a more general class of system: the *affine control systems*, which have the form

$$\frac{dg}{dt} = (\mathcal{X} + Y)(g) + \sum_{j=1}^{m} u_j(\mathcal{X}_j + Y_j)(g),$$

where $\mathcal{X}$, $\mathcal{X}_1, \ldots, \mathcal{X}_m$ are linear and $Y$, $Y_1, \ldots, Y_m$ are right invariant vector fields on $G$. Following the same methodology of the previous chapter, the first step to describe the trajectories is to define a representation from the Euclidean space $\mathbb{R}^{m+1}$ into the group of automorphisms $\mathrm{Aut}(G)$. This is done through the mapping $\rho(t_0, t_1, \ldots, t_m) = \varphi_{t_0} \circ \varphi^1_{t_1} \circ \cdots \circ \varphi^m_{t_m}$, where $\varphi^i_{t_i}$ is the linear flow associated to $\mathcal{X}_i$, $i = 0, 1, \ldots, m$. However, to ensure that this function is truly a representation, we need to assume that the condition

$$[\mathcal{X}_i, \mathcal{X}_j] = 0, \quad \text{for} \quad i, j = 0, 1, \ldots, m$$

holds. In this work, Affine Control Systems satisfying the above assumption are called *Commutative Affine Systems*. We note that controllability is an extremely rare property for this kind of system (see for example [3] or [15]). Nevertheless, the controllability problem is also considered for the inner derivation case. We extend two results from [12] relating the controllability of the commutative affine system to the controllability of an associated right invariant one (Theorems (3.10) and (3.12)). The last section ends with applications on the study of semiconjugations of systems by Lie group homomorphisms.

Unlike dynamical systems, which associate to each point of a differentiable manifold a single trajectory, control systems allow multiple trajectories to depart from the same point. More than that: the path of a trajectory can be altered over time by a change in the control function. This makes the behavior of the trajectories of a control system much more complex to predict - and consequently more chaotic - than that of a dynamic system. Another remarkable difference occurs in the existence of fixed points. For the reasons described above, the incidence of fixed points in control systems is considerably lower than in dynamic systems. For this reason, instead of fixed points it is more convenient for us to consider *invariant sets*. Chapter 4 is an initial proposal to the study of geometric and dynamic properties of control systems on Lie groups. We use the description for the solutions obtained in the previous chapters to show that under certain reasonable conditions these systems solutions are, in a sense, well predictable.

Chapter 1

# Basic Concepts Of Control Theory

This chapter in intended to establish the concepts and notations of control theory necessary to the development of this work. Section 1.1 presents initial definitions and basic results concerning control theory on differentiable manifolds. Section 1.2 deals briefly with the study of linear vector fields, which play an important role in the definition of the most classic control systems on Lie groups. These and others topics related to these subjects can be found in [1], [10], [12], [14] and [18].

## 1.1 Control Systems on Differentiable Manifolds

Let $M$ be a differentiable manifold. A control system on $M$ is defined in terms of an application $X\colon M \times U \to TM$, where for each $u \in U \subset \mathbb{R}^m$ we have that $X(\cdot, u)$ is a smooth vector field. If $u\colon \mathbb{R} \to U$ is a locally integrable function, called admissible control function, then we associate the differential equation

$$\frac{dx}{dt} = X(x(t), u(t)). \tag{1.1}$$

A control system on $M$ is a family $\Sigma$ of differential equations (1.1) parametrized by the admissible control functions $u$. We consider that $u$ belongs to a set $\mathcal{U}$ of locally integrable functions closed under concatenation. That is, given $u$, $v \in \mathcal{U}$ and $T > 0$ the control function defined as $(u \wedge v)(t) = u(t)$ for $t \leq T$, and $(u \wedge v)(t) = v(t - T)$ for $t > T$ also belongs to $\mathcal{U}$.

We assume that for each $x \in X$ and $u\colon \mathbb{R} \to U$, there is a solution $\phi(t, x, u)$ of equation (1.1) with initial condition $\phi(0, x, u) = x$. This makes possible to define a map

$$\phi\colon \mathbb{R} \times M \times \mathcal{U} \to M, \qquad (t, x, u) \to \phi(t, x, u).$$

For each function $u$, the map $\phi_u\colon \mathbb{R}\times M \to M$ is continuous and for $t\in\mathbb{R}$, $\phi_{t,u}\colon M\to M$ is a homeomorphism.

**Example 1.1.** *Linear systems on $\mathbb{R}^n$ are examples of control systems. They are given by the differential equation*

$$\frac{dx}{dt} = Ax + Bu, \tag{1.2}$$

*where $x\in\mathbb{R}^n$, $A$ is a $n\times n$ - matrix, $B$ is a $n\times m$ - matrix, and $u=(u_1,u_2,\ldots,u_m)$ is an admissible control. In this case, the solutions of the system at a point $x_0$ are (see for instance ([9], page 356)*

$$\phi(t,x_0,u) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds.$$

On $\mathcal{U}$ is defined the shift control function $\theta\colon \mathbb{R}\times\mathcal{U}\to\mathcal{U}$, where the control function $\theta(s,u)=\theta_s u$ is defined as

$$(\theta_s u)(t) = u(s+t).$$

This function defines a dynamical system on $\mathcal{U}$ and the solutions of the system (1.1) satisfy what is called the cocycle property:

$$\phi(t+s,x,u) = \phi(t,\phi(s,x,u),\theta_s u).$$

We say that a point $y\in M$ is reachable from $x\in M$ in time $T>0$ if there is a control function $u$ such that $\varphi(T,x,u)=y$. The reachable set (or attainable set) of control system (1.1) from the point $x\in M$ for a time $T>0$ is the set $\mathcal{A}_T(x)$ of all points $y\in M$ reachable from $x$ in time $T>0$. Lastly, we define a larger set: the orbit of a point $x\in M$, denoted as $\mathcal{O}(x)\subset M$, is the set of all points $\phi(t,x,u)$, $t\in\mathbb{R}$ and $u\in\mathcal{U}$. In an orbit, we are allowed to move along the trajectories of the system both forward and backward in time, while in the reachable sets only forward motion is permitted.

The reachable set of a control system may change depending on the set of admissible control functions considered. In this work, we assume that $\mathcal{U}$ is the set of piecewise constant control functions $u\colon\mathbb{R}\to U$, with $0\in\operatorname{int}U\subset\mathbb{R}^m$, defined up next.

**Definition 1.1.** Let $u_1$, $u_2$, $\ldots$, $u_n\in U\subset\mathbb{R}$, $0=t_0<t_1<\cdots<t_n$ positive real

constants and $T = \sum t_i$. A piecewise constant function $u\colon [0,T] \to U$ is defined by

$$u(t) = u_k, \text{ if } \sum_{i=0}^{k-1} t_i < t \leq \sum_{i=0}^{k} t_i.$$

In other words, a piecewise constant function $u\colon [0,T] \to U$ is a function whose domain was split in subintervals in such a way that $u$ is constant in each of these subintervals. The set $U \subset \mathbb{R}^m$ is called the control range of the system. When it is compact and convex, we say that the system is bounded. Otherwise, if $U = \mathbb{R}^m$, the system is said to be unbounded.

Control system (1.1) is said to be controllable from a point $x \in M$ if $\mathcal{A}(x) = M$. Generally, the system is said to be controllable if it is controllable from all points.

In the study of topological properties of orbits and reachable sets it is important to consider the family $\mathcal{F}$ composed of the vector fields $X_u = X(\cdot, u)$. This family gives rise to a distribution on $M$ associating to each $x \in M$ the subspace $\mathrm{Lie}_x \mathcal{F} \subset T_xM$[1]. We say that the system is *full-rank*[2] or satisfies the *accessibility rank condition* if

$$\mathrm{Lie}_x \mathcal{F} = T_xM,$$

for all $x \in M$. About the study of the topological properties of orbits we present what is considered one of the most relevant results due to Nagano - Sussmann (see [1, Theorem 5.1] on page 61):

**Theorem 1.2 (Orbit Theorem).** *Let $\Sigma$ be a control system on a connected smooth manifold $M$ and $x \in M$. Then $\mathcal{O}(x)$ is a connected immersed submanifold of $M$.*

Furthermore, for full-rank systems holds the following important statements:

**Corollary 1.3 (Rashevsky - Chow).** *Let $\Sigma$ be a full-rank system on a connected smooth manifold $M$. Then $\mathcal{O}(x) = M$ for all $x \in M$.*

**Proof:** See Theorem 5.2 of [1], page 65. □

[1] $\mathrm{Lie}_x \mathcal{F}$ is the subspace of all vector fields $X_u(x) = X(x,u) \in T_xM$, with $u \in U$.

[2] Other terms also appear in the literature such as *completely non-holonomic* or *bracket-generating* (see Definition 5.2 in [1] on page 65).

**Theorem 1.4 (Krener).** *If the control system is full-rank, then* $\mathcal{A}(x) \subset \overline{\operatorname{int} \mathcal{A}(x)}$*, for all* $x \in M$*. In particular, reachable sets for arbitrary time have nonempty interior.*

**Proof:** See Theorem 8.1 of [1], page 107. □

The above result means that in full-rank systems, both reachable sets and orbits are full-dimensional. Another consequence, as the next corollary shows, is the simplification of the study of controllability since it allows us to look more generally the closure of reachable sets.

**Corollary 1.5.** *Suppose that a control system is full-rank. If for some* $x \in M$ *we have* $\overline{\mathcal{A}(x)} = M$ *then* $\mathcal{A}(x) = M$*. That is, the system is controllable from* $x$*.*

**Proof:** See Corollary 8.1 of [1], page 110. □

In the next chapters, we will be interested in the study of some classes of control systems on Lie groups. These systems to be defined are particular cases of *Affine Control System*. Therefore, it is convenient to give the following definition:

**Definition 1.6.** *An affine control system on* $M$ *is a family of differential equations of the form*

$$\frac{dx}{dt} = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)). \tag{1.3}$$

*where* $X_0, X_1, \ldots, X_m$ *are smooth vector fields on* $M$*,* $u = (u_1, \ldots, u_m)\colon \mathbb{R} \to U$ *is a piecewise constant control function and* $U \subset \mathbb{R}^m$ *is compact, convex and such that* $0 \in \operatorname{int} U$*.*

Among other things, the importance of this system is in the simplicity of its solutions. As we show next, it is possible to obtain all the solutions of the system just concatenating the solutions associated to constant control functions.

**Proposition 1.7.** *Let* $u_1, u_2, \ldots, u_n$ *be real constants,* $0 = t_0 < t_1 < \cdots < t_n$*,* $T = \sum t_i$ *and* $u\colon [0, T] \to U$ *be a piecewise constant control function defined by*

$$u(t) = u_k, \;\; \textit{if} \;\; \sum_{i=0}^{k-1} t_i < t \leq \sum_{i=0}^{k} t_i,$$

*for* $k = 1, \ldots, n$. *Then*

$$\phi(t, x, u) = \phi_{t_n}(u_n, \ldots (\phi_{t_2}(u_2, \phi_{t_1}(u_1, x)))).$$

**Proof:** A proof can be found in [28, Proposition 1.1.3]. □

We end up this section noting that for Affine Control Systems, the accessibility rank condition can be easier checked simply observing that $\text{Lie}(\mathcal{F}) = \text{Lie}(X_0, X_1, \ldots, X_m)$.

## 1.2 Linear Vector Fields on Lie Groups

This section is intended to recall some important facts about linear vector fields on Lie groups and their flows. These concepts are necessary to introduce some of the most important classes of control systems such as linear, bilinear and affine systems. These systems will be the main object of study in the chapters to come. For more details about the following concepts see [13] and [18].

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Throughout this work, $\mathfrak{g}$ is the set of right invariant vector fields. For every $g \in G$, the maps $R_g, L_g \colon G \to G$ are, respectively, the right and left translations on $G$.

**Definition 1.8.** *The normalizer of the Lie algebra* $\mathfrak{g}$ *is the set of all vector fields* $\mathcal{F}$ *such that* $[\mathcal{F}, Y] \in \mathfrak{g}$, *for all* $Y \in \mathfrak{g}$.

The vector fields $\mathcal{F}$ are called affine. The set of all affine vector fields is denoted by $Norm(\mathfrak{g})$. This set is actually a Lie algebra since the Lie bracket $[\mathcal{F}_1, \mathcal{F}_2]$ of two affine system is still an affine vector field. Proposition 3.3 of [13] in page 962 shows that affine vector fields are complete.

**Definition 1.9.** *A vector field* $\mathcal{X}$ *on* $G$ *is said to be linear if* $\mathcal{X} \in Norm(\mathfrak{g})$ *and* $\mathcal{X}(e) = 0$.

Denoting by $\varphi_t$ the flow of a linear vector field, it follows from the definition that $\varphi_t(e) = e$, for all $t \in \mathbb{R}$. The set of all linear vector fields is denoted by $\mathcal{L}(\mathfrak{g})$. A direct calculation shows that it is a subalgebra of $Norm(\mathfrak{g})$.

The word affine used to refer to $\mathcal{F}$ is due to the fact that affine vector fields can be uniquely decomposed as a sum of a linear vector field and a right invariant one. This is the subject of the two next results.

**Proposition 1.10.** *The mapping* ad: $Norm(\mathfrak{g}) \to Der(\mathfrak{g})$ *defined by* $\mathcal{F} \to \mathrm{ad}(\mathcal{F})$ *is a Lie algebra homomorphism.*

**Proof:** See [18, Proposition 1.18], page 22. □

**Proposition 1.11.** *The kernel of the mapping* $\mathcal{F} \to \mathrm{ad}(\mathcal{F})$ *is the set of left invariant vector fields. An affine vector field* $\mathcal{F}$ *can be uniquely decomposed into a sum*

$$\mathcal{F} = \mathcal{X} + Y$$

*where* $\mathcal{X}$ *is linear and* $Y$ *is right invariant.*

**Proof:** See [13, Proposition 3.1], page 959. □

The most important fact about linear vector fields is that its flow is an one-parameter group of automorphisms. More precisely, in [13, Theorem 3.1] page 959, it is showed that a linear vector field $\mathcal{X}$ can be characterized by any of the following equivalent conditions:

(i) for all $t \in \mathbb{R}$, $\varphi_t$ is an automorphism of $G$;

(ii) for all $Y \in \mathfrak{g}$, $[\mathcal{X}, Y] \in \mathfrak{g}$ and $\mathcal{X}(e) = 0$, where $e$ is identity of $G$.

(iii) for all $g, h \in G$, $\mathcal{X}(gh) = d(R_h)_g\mathcal{X}(g) + d(L_g)_h\mathcal{X}(h)$.

**Example 1.2.** *In* $\mathbb{R}^n$*, linear vector fields are given by real matrices* $A \in M(n \times n; \mathbb{R})$*, since their flows* $e^{tA}$ *are one-parameter groups of automorphisms.*

**Example 1.3.** *Consider* $G = Gl(n; \mathbb{R})$ *the group of all* $n \times n$ *non-singular matrices. Given* $A \in M(n \times n; \mathbb{R})$*, the vector field defined as*

$$\mathcal{X}(M) = AM - MA, \quad M \in Gl(n; \mathbb{R}),$$

*is linear. Its flows is given by* $\varphi_t(M) = e^{tA}Me^{-tA}$*.*

For a linear vector field $\mathcal{X}$, it is possible to associate a derivation $\mathcal{D}\colon \mathfrak{g} \to \mathfrak{g}$ defined by $\mathcal{D}(Y) = -[\mathcal{X}, Y]$. For the derivation $\mathcal{D}$ and the linear flow $\varphi$ are valid the following properties (See [13, Proposition (3.2)], on page 961)

$$d(\varphi_t)_e = e^{t\mathcal{D}} \qquad \text{and} \qquad \varphi_t(\exp(Y)) = \exp(e^{t\mathcal{D}}Y).$$

A particular case of derivations is the inner derivation. That is, $\mathcal{D} = -ad(X)$ for some $X \in \mathfrak{g}$. In this case, the linear vector field $\mathcal{X}$ can be decomposed as $\mathcal{X} = X + dIX$, where $dIX$ is the left invariant vector field induced by the Inverse Map $I(g) = g^{-1}$. Conversely, for any $X \in \mathfrak{g}$, the vector field defined by $\mathcal{X} = X + dIX$ is linear.

Of course, not all derivations on a Lie group $G$ is associated to a linear vector field. However, the following result holds:

**Proposition 1.12.** *Let $G$ be a simply connected Lie group and $\mathcal{D}$ a derivation of its Lie algebra. Then there exists only one linear vector field whose associated derivation is $\mathcal{D}$.*

Chapter 2

# Solution Curve For Linear Control Systems

Let $G$ be a connected Lie group. In this chapter we intend to give explicitly a description for the solution curve at the identity of Linear Control Systems on $G$, denoted here as $\Sigma_L$. In order to do that, we use a construction presented in [9] that associates to $\Sigma_L$ an Invariant Control System defined on a semidirect product $G \times_\varphi \mathbb{R}$, where $\varphi$ is the flow of the linear vector field of $\Sigma_L$. The definition of linear system that we give below was introduced by Ayala and Tirao in [5]. It represents the natural extension of the same system defined for Euclidean spaces.

**Definition 2.1.** *Let $G$ be a connected Lie group. A linear control system on $G$ is a family of differential equations of the form*

$$\frac{dg}{dt} = \mathcal{X}(g) + \sum_{i=1}^{m} u_i(t) Y_i(g),$$

*where $g \in G$, $\mathcal{X}$ is a linear vector field, $Y_1, \ldots, Y_m$ are right invariant vector fields, and $u = (u_1, \ldots, u_m) \colon U \to \mathbb{R}^m$ is an admissible control function.*

Before we move on to the main purpose of this chapter, we would like to present some important properties of solutions and reachable sets for linear systems that are used throughout this work. The results below are based on [12].

**Proposition 2.2.** *Let $g \in G$ and $u$ be an admissible control function. Then*

$$\phi(t, g, u) = \phi(t, e, u) \cdot \varphi_t(g),$$

*for $t \in \mathbb{R}$.*

**Proposition 2.3.** *Consider a linear control system defined on a Lie group $G$. With respect to reachable sets, the following assertions hold:*

*1)* $\mathcal{A}_s(e) \subset \mathcal{A}_t(e)$, *for all* $0 \leq s \leq t$.

*2)* $\mathcal{A}_t(g) = \mathcal{A}_t(e)\varphi_t(g)$, *for all* $t > 0$ *and* $g \in G$.

*3)* $\mathcal{A}_{t+s}(e) = \mathcal{A}_t(e)\varphi_t(\mathcal{A}_s(e))$, *for all* $s, t \geq 0$.

*Proof.* The proof of 1) can be found in [27], while 2) and 3) are direct consequences of Proposition 2.2. □

Moving on to the main purpose of this chapter, we begin recalling that the flow $\varphi_t$ of the linear vector field $\mathcal{X}$ yields a representation $\varphi\colon \mathbb{R} \to Aut(G)$. This fact allows us to define the semidirect product $G \times_\varphi \mathbb{R}$, that is, the set $G \times \mathbb{R}$ endowed with the product $(g, t)(h, s) = (g\varphi_t(h), t + s)$ (see [22, Section 9.3]).

It is well-known that $G \times_\varphi \mathbb{R}$ is a Lie group. Furthermore, its correspondent Lie algebra is the semidirect product of Lie algebras $\mathfrak{g} \times_\sigma \mathbb{R}$, where $\sigma\colon \mathbb{R} \to Der(\mathfrak{g})$ is defined as

$$\sigma(t)(Y) = ad_{t\mathcal{X}}(Y) = t[\mathcal{X}, Y],$$

for all $Y \in \mathfrak{g}$. The relation between $\varphi$ and $\sigma$ is given by $d\varphi_0 = \sigma$.

For any vector fields $(Y, t), (W, s) \in \mathfrak{g} \times_\sigma \mathbb{R}$, the Lie bracket is given by the formula

$$[(Y, t), (W, s)] = ([Y, W] + \sigma(t)(W) - \sigma(s)(Y), [t, s]) = ([Y + t\mathcal{X}, W + s\mathcal{X}], 0)\,[1].$$

Throughout this work, for $(g, r) \in G \times_\varphi \mathbb{R}$, $R_{(g,r)}$ denotes the right translation and, when not specified, the differential $dR_{(g,r)}$ is evaluated at the group identity.

Now we determine the value of an invariant vector field $(W, s)$ on an arbitrary point $(g, r)$.

**Proposition 2.4.** *If* $(W, s) \in \mathfrak{g} \times_\sigma \mathbb{R}$ *and* $(g, r) \in G \times_\varphi \mathbb{R}$, *then*

$$(W, s)(g, r) = (W(g) + s\mathcal{X}(g), s)\,.$$

[1]It follows directly from Definition 9.9 of [22, Section 9.3], page 194

**Proof:** We first write $(W,s)(g,r) = dR_{(g,r)}(W,s)$. Thus, in matrix notation, the differential $dR_{(g,r)}$ gives

$$dR_{(g,r)}(W,s) = \begin{pmatrix} dR_g & \mathcal{X}(g) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W \\ s \end{pmatrix} = (W(g) + s\mathcal{X}(g), s)\,.$$

□

This proposition allows us to describe exponential curves of invariant vector fields of $\mathfrak{g} \times_\sigma \mathbb{R}$.

**Lemma 2.5.** *If $(W,0),(0,s) \in \mathfrak{g} \times_\sigma \mathbb{R}$, then their exponentials are the smooth curves $(\exp(tW),0)$ and $(e,st)$, respectively, with $t \in \mathbb{R}$.*

**Proof:** We first compute the exponential for $(W,0)$. To this purpose, we note that $(W,0)(g,r) = (W(g),0)$ for all $(g,r) \in G \times_\varphi \mathbb{R}$ in view of the Proposition 2.4. By definition of exponential,

$$\frac{d}{dt}(\exp(tW),0) = \left(\frac{d}{dt}\exp(tW),0\right) = (W(\exp(tW)),0) = (W,0)(\exp(tW),0).$$

The result follows by uniqueness of solution. Analogously we can see that the curve $(e,st)$ is the exponential curve of $(0,s)$. □

In the following, the previous lemma will be used to determine the exponential of an invariant vector field $(W,s) \in \mathfrak{g} \times_\sigma \mathbb{R}$.

**Proposition 2.6.** *Let $(W,s)$ be an invariant vector field on $G \times_\sigma \mathbb{R}$. It follows that*

$$\exp(t(W,s)) = \left(\lim_{n\to\infty} \prod_{i=0}^{n-1} \varphi_{ist/n} \circ \exp(t/n \cdot W), st\right). \tag{2.1}$$

**Proof:** We first write $(W,s) = (W,0) + (0,s)$. Applying the Lie product formula (Proposition (6.11) of [22], page 133) gives

$$\exp(t(W,s)) \quad = \quad \lim_{n\to\infty} \left(\exp(t/n(W,0)) \cdot \exp(t/n(0,s))\right)^n.$$

Using the above lemma and the semidirect product we see that

$$\begin{aligned}\exp(t(W,s)) &= \lim_{n\to\infty}((\exp(t/n\cdot W),0)(e,st/n))^n\\ &= \lim_{n\to\infty}(\exp(t/n\cdot W),st/n)^n\\ &= \left(\lim_{n\to\infty}\prod_{i=0}^{n-1}\varphi_{ist/n}\circ\exp(t/n\cdot W),st\right),\end{aligned}$$

and the proof is complete. □

Using the relation $d(\varphi_t)_e = e^{t\mathcal{D}}$ we can rewrite formula (2.1) as

$$\exp(t(W,s)) = \left(\lim_{n\to\infty}\prod_{i=0}^{n-1}\exp\left(t/n\cdot e^{\mathcal{D}_t}W\right),st\right), \tag{2.2}$$

where, for simplification, we denote $\mathcal{D}_t = \dfrac{ist}{n}\mathcal{D}$.

Now consider the vector fields $\bar{\mathcal{X}} = (0,1)$, $\bar{Y_j} = (Y_j,0) \in \mathfrak{g}\times_\sigma\mathbb{R}$, for each $j = 1,\ldots,m$. It follows from Proposition 2.4 that, in coordinates, these vector fields can be expressed as

$$\bar{\mathcal{X}}(g,r) = (\mathcal{X}(g),1) \qquad \text{and} \qquad \bar{Y_j}(g,r) = (Y_j(g),0).$$

We define the following invariant control system on $G\times_\varphi\mathbb{R}$:

$$\Sigma_I\colon \frac{d(g,r)}{dt} = \bar{\mathcal{X}}(g,r) + \sum_{j=1}^m u_j\bar{Y_j}(g,r).$$

Equivalently, in coordinates we have

$$\begin{pmatrix} dg/dt \\ dr/dt\end{pmatrix} = \begin{pmatrix}\mathcal{X}(g) + \sum_{j=1}^m u_jY_j(g)\\ 1\end{pmatrix}.$$

The invariant system above was built in such a way that $\pi(\Sigma_I) = \Sigma_L$, where $\pi\colon G\times_\varphi\mathbb{R}\to G$ is the projection on the first coordinate. We explain the meaning of this notation as follows: if we denote $\mathcal{A}_I(g,r)$ the reachable set of $\Sigma_I$ at point $(g,r)$ and $\mathcal{A}_L(g)$ the reachable set of $\Sigma_L$ at point $g$, then $\pi(\mathcal{A}_I(g,r)) = \mathcal{A}_L(g)$. Furthermore, the invariance of the system allows us to write $\mathcal{A}_I(g,r) = \mathcal{A}_I(e,0)\cdot(g,r)$. And now, we move on to the main result.

**Theorem 2.7.** *For $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ the curve*

$$\phi(t,e,u) = \lim_{n\to\infty} \prod_{i=0}^{n-1} \varphi_{it/n} \circ \exp\left(\frac{t}{n}\sum_{j=1}^{m} u_j Y_j\right) \tag{2.3}$$

*is the solution, with initial condition $\phi(0,e,u) = e$, of the linear dynamical system*

$$\Sigma_L\colon \frac{dg}{dt} = \mathcal{X}(g) + \sum_{j=1}^{m} u_j Y_j(g).$$

**Proof:** Let us denote $W = \sum_{j=1}^{m} u_j Y_j$ and $\exp(t(W,1)) = (\phi(t,e,u),t)$. From Proposition 2.4 we have that

$$(W,1)(\phi(t,e,u),t) = (W(\phi(t,e,u) + \mathcal{X}(\phi(t,e,u)),1) = \left(\mathcal{X}(\phi(t,e,u)) + \sum_{j=1}^{m} u_j Y_j(\phi(t,e,u)),1\right).$$

On the other hand, the curve $(\phi(t,e,u),t)$ is the integral curve of $W$. Therefore

$$(W,1)\exp(t(W,1)) = (W,1)(\phi(t,e,u),t) = \left(\frac{d}{dt}\phi(t,e,u),1\right).$$

We thus get

$$\begin{pmatrix} \frac{d}{dt}\phi(t,e,u) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{X}(\phi(t,e,u)) + \sum_{j=1}^{m} u_j Y_j(\phi(t,e,u)) \\ 1 \end{pmatrix}.$$

Taking the projection on the first coordinate we conclude that the curve $\phi(t,e,u)$ satisfies the differential equation of the dynamical system. As $\phi(0,e,u) = e$, we have that this is the solution of the system at the identity. Besides that, Proposition 2.6 give us a description of $\exp(t(W,1))$. Since $\exp(t(W,1)) = (\phi(t,e,u),t)$, we conclude that

$$\phi(t,e,u) = \lim_{n\to\infty} \prod_{i=0}^{n-1} \varphi_{it/n} \circ \exp\left(\frac{t}{n}\sum_{j=1}^{m} u_j Y_j\right).$$

□

Applying the above theorem it is possible to recall a well-known result about linear systems:

**Corollary 2.8.** *Let $g \in G$ be an arbitrary point and consider the linear dynamical system as in the previous theorem. The solution $\phi(t, e, u)$ of the system starting at an arbitrary point $g$ is given by the formula*

$$\phi(t, g, u) = \phi(t, e, u)\varphi_t(g).$$

**Proof:** Consider an arbitrary point $(g, r) \in G \times_\varphi \mathbb{R}$. Denote by $\phi_I((g, r), t)$ the solution of the control system $\Sigma_I$ at $(g, r)$. We have already seen that $\phi_L(g, t) = \pi(\phi_I((g, r), t))$. Now, using the right invariance property and the product on $G \times_\varphi \mathbb{R}$, we get

$$\phi(t, g, u) = \pi\left(\phi_I((e, 0), t)(g, r)\right).$$

By the proof of the previous theorem, it follows that $\phi_I((e, 0), t) = (\phi_t(e, u), t)$. Therefore

$$\phi_t(g, u) = \pi(\phi_t(e, u)\varphi_t(g), t + r) = \phi(t, e, u)\varphi_t(g).$$

□

**Example 2.1** (**Linear Control Systems on Heisenberg Group).** Let $G$ be the *Heisenberg* group, that is, the set of all real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

As usual, we identify this group with $\mathbb{R}^3$ in such a way that the group product is defined as

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2).$$

In this case, the Lie algebra of the Heisenberg group is the vector space $\mathbb{R}^3$ with the Lie bracket defined as $[(x_1, y_1, z_1), (x_2, y_2, z_2)] = (0, 0, x_1y_2 - x_2y_1)$ and the exponential map $\exp\colon \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\exp(x, y, z) = \left(x, y, \frac{xy}{2} + z\right).$$

The right invariant vector fields $Y = (m, n, p)$ in $G$ have the form

$$Y(x, y, z) = m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y} + (my + p)\frac{\partial}{\partial z},$$

while the matrix of a derivation $\mathcal{D}$ associated to a linear vector field $\mathcal{X}$ is written as

$$\mathcal{D} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} + a_{22} \end{pmatrix}.$$

We consider the following linear control system on $G$

$$\frac{dg}{dt} = \mathcal{X}(g) + uY(g),$$

where $Y = (0, 0, p)$ and $\mathcal{X}$ is the linear vector field associated to the derivation

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying formula (2.2), we note by a direct calculation that

$$\frac{ut}{n} e^{it/n \cdot \mathcal{D}} Y = \left(0, 0, \frac{upt}{n} \cdot e^{it/n}\right).$$

Taking the exponential and the limit we obtain

$$\phi(t, e, u) = \left(0, 0, \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{upt}{n} \cdot e^{it/n}\right) = \left(0, 0, upt \int_0^t e^s ds\right) = \left(0, 0, e^t \cdot upt\right).$$

From the solution above it is easy to see that the linear control system is not controllable from the identity.

**Example 2.2 (Linear control system on $\boldsymbol{Gl(n;\mathbb{R})^+}$).** Let $Gl(n;\mathbb{R})^+$ be the set of all $n \times n$ real matrices with positive determinant and $\mathfrak{gl}(n;\mathbb{R})$ its Lie algebra. As mentioned before, for $A \in \mathfrak{gl}(n;\mathbb{R})$ the vector field $\mathcal{X}_A(g) = Ag - gA$ is linear, and its linear flow is given by $\varphi_t(g) = e^{tA}ge^{-tA}$. Consider the right invariant vector fields $B_1, \ldots, B_m \in \mathfrak{gl}(n;\mathbb{R})$ defined by $B_j(g) = B_j g$. Define a linear control system on $Gl(n;\mathbb{R})^+$ by

$$\frac{dg}{dt} = Ag - gA + \sum_{j=1}^{m} u_j B_j(g). \tag{2.4}$$

Applying formula (2.3) to find the solution of the linear control system above, we obtain

$$\phi(t, e, u) = e^{t(A+\sum u_j Bj)}e^{-tA}.$$

Chapter 3

# Solution Curve For Affine Control Systems

Motivated by the results obtained in the previous chapter, our intention now is to study a more general control system on $G$: The Affine Control System, which is a system of the form

$$\Sigma_A\colon \frac{dg}{dt} = (\mathcal{X} + Y)(g) + \sum_{i=1}^{m} u_i(\mathcal{X}_i + Y_i)(g), \tag{3.1}$$

where $\mathcal{X}$, $\mathcal{X}_1$, ..., $\mathcal{X}_m$ are linear vector fields, $Y$, $Y_1$, ..., $Y_m$ are right invariant vector fields, and $u = (u_1, \dots, u_m)$ is an admissible control. This system is a natural extension of Linear Control Systems and have been an object of study of many authors including Jouan in [13], Jurjdevic and Sallet in [15], Kara and San Martin in [17], Rocio, Santana and Verdi in [20] and more recently, Ayala, da Silva and Ferreira in [3].

Our first step is to describe the solutions of the system and later apply them on the study of the inner derivation case and system conjugation. However, following the same techniques and procedures of the previous chapter, a handicap comes up: for linear systems, we defined a representation $\varphi\colon \mathbb{R} \to Aut(G)$ in terms of the flow of the linear vector field $\mathcal{X}$. How can we extend this idea considering that $m+1$ linear vector fields appear in the definition of the affine system 3.1?

A natural answer to this question is to consider the application $\rho\colon \mathbb{R}^{m+1} \to Aut(G)$ defined as

$$\rho(t_0, t_1, \dots, t_m) = \varphi_{t_0} \circ \varphi^1_{t_1} \circ \cdots \circ \varphi^m_{t_m}$$

where $\varphi_{t_0}$, $\varphi^1_{t_1}$, ..., $\varphi^m_{t_m}$ are the linear flows associated to the linear vector fields $\mathcal{X} =$

$\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_m$, respectively.

In order for $\rho$ to become a representation of the Euclidean space $\mathbb{R}^m$ into $Aut(G)$, we need to assume all along this work that

$$[\mathcal{X}_i, \mathcal{X}_j] = 0, \quad \text{for} \quad i, j = 0, 1, \ldots, m. \tag{3.2}$$

In this work, Affine Control Systems satisfying the assumption (3.2) are called *Commutative Affine Systems*. This assumption, as we will see, can be found naturally in semi simple Lie groups and in direct products. With this assumption in hands, following a procedure analogous to that used in the previous chapter, we describe the solution for the system (3.1).

A second step is to consider the case of inner derivations. This approach has already been used in [12] in the context of linear systems. We extend the study for affine control systems and, as expected, the solution curve gets a simpler description. Furthermore, we can prove that an invariant control system $\Sigma_I$ is naturally associated and, under certain conditions, its controllability is closely related to the controllability of $\Sigma_A$.

The chapter ends with a study of conjugation between affine systems. Our idea is based in a conjugation by homomorphism of linear system. We show that a necessary and sufficient condition for two affine systems to be conjugate is that the flows of linear vector fields are semi conjugate and the invariant vector fields are related in a sense that will be established in Theorem (3.14).

## 3.1 Solution for Commutative Affine Control Systems

In this section, we wish to describe the solution curve of Affine Control System on Lie groups. In this first moment, the same methodology of the previous chapter is applied. Let $G$ be a connected Lie group. Consider a Commutative Control System on $G$:

$$\Sigma_A \colon \frac{dg}{dt} = (\mathcal{X} + Y)(g) + \sum_{i=1}^{m} u_i(\mathcal{X}_i + Y_i)(g). \tag{3.3}$$

Let us denote by $\varphi_{t_0}$, $\varphi^1_{t_1}$, ..., $\varphi^m_{t_m}$ the linear flows associated to the linear vector fields $\mathcal{X} = \mathcal{X}_0$, $\mathcal{X}_1$, ..., $\mathcal{X}_m$[1], respectively. We define the application $\rho\colon \mathbb{R}^{m+1} \to Aut(G)$,

$$\rho(t_0, t_1, \ldots, t_m) = \varphi_{t_0} \circ \varphi^1_{t_1} \circ \cdots \circ \varphi^m_{t_m}.$$

For convenience, given $t = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1}$, we will write $\rho_t$ instead of $\rho(t_0, t_1, \ldots, t_m)$.

Under the condition

$$[\mathcal{X}_i, \mathcal{X}_j] = 0, \quad \text{for} \quad i, j = 0,\, 1,\, \ldots,\, m, \tag{3.4}$$

the application $\rho$ is a representation of the Euclidean Space $\mathbb{R}^{m+1}$ into $G$. In fact,

$$\begin{aligned}
\rho(t_0 + s_0, t_1 + s_1, \ldots, t_m + s_m) &= \varphi_{t_0+s_0} \circ \varphi^1_{t_1+s_1} \circ \cdots \circ \varphi^m_{t_m+s_m} \\
&= \varphi_{t_0} \circ \varphi_{s_0} \circ \varphi^1_{t_1} \circ \varphi^1_{s_1} \circ \cdots \circ \varphi^m_{t_m} \circ \varphi^m_{s_m} \\
&= \varphi_{t_0} \circ \varphi^1_{t_1} \circ \cdots \circ \varphi^m_{t_m} \circ \varphi_{s_0} \circ \varphi^1_{s_1} \circ \cdots \circ \varphi^m_{s_m} \\
&= \rho(t_0, t_1, \ldots, t_m) \circ \rho(s_0, s_1, \ldots, s_m).
\end{aligned}$$

**Remark 3.1.** The assumption (3.4) can be found, for example, in a direct product $G = G_0 \times G_1 \times \ldots \times G_m$. Taking each $\mathcal{X}_i \in G_i$, we can view that it is automatically satisfied. An especial case of this is when $G$ is a compact Lie group because it is isomorphic to a direct product of simple, compact, connected and simply connected Lie groups. Or yet, consider a semi simple Lie group $G$ with Lie algebra

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m,$$

where $\mathfrak{g}_i$ are its simple components. We can define an affine system on $G$ taking $X_i \in \mathfrak{g}_i$ and setting $\mathcal{X}_i = X_i + dIX_i$. It follows that $[\mathcal{X}_i, \mathcal{X}_j] = 0$ because $[X_i, X_j] = 0$.

We define the semidirect product $G \times_\rho \mathbb{R}^{m+1}$. That is, the cartesian product of $G$ and $\mathbb{R}^{m+1}$ endowed with the product $(g, t)(h, s) = (g\rho_t(h), t + s)$. This set is a Lie group and the correspondent Lie algebra is the semidirect product of algebras $\mathfrak{g} \times_\sigma \mathbb{R}^{m+1}$, where

[1] All over the work, we consider always $\mathcal{X}_0 = \mathcal{X}$ and $Y_0 = Y$.

$\sigma\colon \mathbb{R}^{m+1} \to Der(\mathfrak{g})$ is defined as

$$\sigma_t(Y) = \sigma(t)(Y) = ad_{\sum t_i \mathcal{X}_i}(Y),$$

for $t = (t_0, \ldots, t_m)$ and $Y \in \mathfrak{g}$.

**Proposition 3.2.** *Let $(Y, t_0, \ldots, t_m)$, $(W, s_0, \ldots, s_m)$ be vector fields in $\mathfrak{g} \times_\sigma \mathbb{R}^{m+1}$. Then*

$$[(Y, t_0, \ldots, t_m), (W, s_0, \ldots, s_m)] = \left( [Y + \sum_{i=0}^{m} t_i \mathcal{X}_i, W + \sum_{j=0}^{m} s_j \mathcal{X}_j], 0, \ldots, 0 \right).$$

**Proof:** We begin by computing

$$\begin{aligned} [(Y, t), (W, s)] &= ([Y, W] + \sigma_t(W) - \sigma_s(Y), [t, s]) \\ &= \left([Y, W] + ad_{\sum t_i \mathcal{X}_i}(W) - ad_{\sum s_j \mathcal{X}_j}(Y), 0\right). \end{aligned}$$

Adding $[\sum t_i \mathcal{X}_i, \sum s_j \mathcal{X}_j] = 0$ in the first coordinate we get

$$[(Y, t), (W, s)] = \left([Y + \sum t_i \mathcal{X}_i, W + \sum s_j \mathcal{X}_j], 0, \ldots, 0\right).$$

□

Let $(W, 0, \ldots, 0)$, $(0, s_0, s_1, \ldots, s_m) \in \mathfrak{g} \times_\sigma \mathbb{R}^{m+1}$. A direct calculation proves that their exponentials are $(\exp(tW), 0, \ldots, 0)$ and $(e, s_0 t, s_1 t, \ldots, s_m t)$, respectively. This fact allows us to obtain the exponential curve for any invariant vector field on $G \times_\rho \mathbb{R}^{m+1}$.

**Proposition 3.3.** *If $(W, s_0, s_1, \ldots, s_m)$ is a vector field in $\mathfrak{g} \times_\sigma \mathbb{R}^{m+1}$, then*

$$\exp(t(W, s)) = \left( \lim_{n \to \infty} \prod_{i=0}^{n-1} \rho(i s_0 t/n, \ldots, i s_m t/n)(\exp(t/n \cdot W)), s_0 t, \ldots, s_m t \right). \quad (3.5)$$

**Proof:** We first write $(W, s_0, \ldots, s_m) = (W, 0, \ldots, 0) + (0, s_0, \ldots, s_m)$. Now, applying the

Lie Product Formula we obtain

$$\begin{aligned}
\exp(t(W,s)) &= \lim_{n\to\infty} \left(\exp(t/n \cdot W, 0) \cdot \exp(0, ts/n)\right)^n \\
&= \lim_{n\to\infty} \left((\exp(t/n \cdot W), 0, \ldots, 0)(e, s_0 t/n, s_1 t/n, \ldots, s_m t/n)\right)^n \\
&= \lim_{n\to\infty} (\exp(t/n \cdot W), s_0 t/n, s_1 t/n, \ldots, s_m t/n)^n \\
&= \left(\lim_{n\to\infty} \prod_{i=0}^{n-1} \rho(i s_0 t/n, \ldots, i s_m t/n)(\exp(t/n \cdot W)), s_0 t, \ldots, s_m t\right).
\end{aligned}$$

□

Denoting by $D_{\mathcal{X}_0}$, $D_{\mathcal{X}_1}$, …, $D_{\mathcal{X}_m}$ the derivations of the linear vector fields $\mathcal{X}_0$, $\mathcal{X}_1$, …, $\mathcal{X}_m$, respectively, we can rewrite the above result as

$$\exp(t(W,s)) = \left(\lim_{n\to\infty} \prod_{i=0}^{n-1} \exp\left(t/n \cdot e^{D_t} W\right), s_0 t, \ldots, s_m t\right) \tag{3.6}$$

where $D_t = \frac{it}{n} D_{\mathcal{X}_0} + \frac{i u_1 t}{n} D_{\mathcal{X}_1} + \cdots + \frac{i u_m t}{n} D_{\mathcal{X}_m}$.

Our next step is to describe the effect of an invariant vector field $(W, s_0, \ldots, s_m)$ on a point $(g, r_0, \ldots, r_m)$.

**Proposition 3.4.** *If $(W,s) \in \mathfrak{g} \times_\sigma \mathbb{R}^{m+1}$ and if $(g,r) \in G \times_\rho \mathbb{R}^{m+1}$, then*

$$(W,s)(g,r) = \left(W(g) + \sum_{i=0}^{m} s_i \mathcal{X}_i(g), s_0, \ldots, s_m\right),$$

*where $s = (s_0, \ldots, s_m)$ and $r = (r_0, \ldots, r_m)$.*

**Proof:** We begin by using the right invariance property. In fact,

$$(W,s)(g,r) = d(R_{(g,r)})(W,s) = d(R_{(g,r)})(W,0) + \sum_{i=0}^{m} d(R_{(g,r)})(0, \ldots, s_i, \ldots, 0).$$

It follows from the definition of exponential curve on $G \times_\rho \mathbb{R}^{m+1}$ that

$$\begin{aligned}(W,s)(g,r) &= \frac{d}{dt}((\exp(tW),0)(g,r))\Big|_{t=0} + \sum \frac{d}{dt}(e,0,\dots,s_i t,\dots,0)(g,r))\Big|_{t=0} \\ &= \frac{d}{dt}(\exp(tW)g,r)\Big|_{t=0} + \sum \frac{d}{dt}(\varphi^i_{s_i t}(g),0,\dots,s_i t + r_i,\dots,0)\Big|_{t=0}.\end{aligned}$$

Differentiating with respect to $t$ each term of the right side yields

$$(W,s)(g,r) = (W(g),0) + \sum_{i=0}^{m}(s_i\mathcal{X}_i(g),0,\dots,s_i,\dots,0) = \left(W(g) + \sum_{i=0}^{m} s_i\mathcal{X}_i(g), s\right).$$

□

Consider the invariant vector fields $\bar{\mathcal{X}}_j = (0,\dots,1,\dots,0)$ and $\bar{Y}_j = (Y_j,0,\dots,0) \in \mathfrak{g} \times_\sigma \mathbb{R}^{m+1}$, for $j = 0,1,\dots,m$, where the 1 stands in the $(j+2)$ - th position. From the previous proposition we see that, in coordinates, these fields can still be expressed as

$$\bar{\mathcal{X}}_j(g,r) = (\mathcal{X}_j(g),0,\dots,1,\dots,0) \qquad \text{and} \qquad \bar{Y}_j(g,r) = (Y_j(g),0,\dots,0),$$

for $j = 0,\dots,m$. By means of these fields we associate to the affine system (3.1) the following invariant control system on $G \times_\rho \mathbb{R}^{m+1}$:

$$\bar{\Sigma}_I\colon \frac{d(g,r)}{dt} = (\bar{\mathcal{X}} + \bar{Y})(g,r) + \sum u_j(\bar{\mathcal{X}}_j + \bar{Y}_j)(g,r).$$

In coordinates, we have

$$\begin{pmatrix} dg/dt \\ dr_0/dt \\ dr_1/dt \\ \vdots \\ dr_m/dt \end{pmatrix} = \begin{pmatrix} (\mathcal{X}+Y)(g) + \sum u_j(\mathcal{X}_j + Y_j)(g) \\ 1 \\ u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

This means that the invariant control system $\bar{\Sigma}_I$ was built to satisfy the condition $\pi(\bar{\Sigma}) = \Sigma$, where $\pi\colon G \times_\rho \mathbb{R}^{m+1} \to G$ is the projection on the first coordinate. As a direct consequence, if we denote $\bar{\mathcal{A}}_T(g,r)$ the reachable set of a point $(g,r)$ in time $T > 0$ of $\bar{\Sigma}_I$, then

$\pi(\bar{\mathcal{A}}_T(g,r)) = \mathcal{A}_T(g)$.

We are now in position to prove our main result.

**Theorem 3.5.** *Consider the curve*

$$\phi(t,e,u) = \lim_{n\to\infty} \prod_{i=0}^{n-1} \rho(it/n, iu_1 t/n, \ldots, iu_m t/n) \exp\left(\frac{t}{n}\sum_{j=1}^{m} u_j Y_j\right), \tag{3.7}$$

*where* $u = (1, u_1, \ldots, u_m) \in \mathbb{R}^{m+1}$. *Then* $\phi(t,e,u)$ *is the solution of the affine dynamical system*

$$\frac{dg}{dt} = (\mathcal{X} + Y)(g) + \sum_{j=1}^{m} u_j(\mathcal{X}_j + Y_j)(g) \tag{3.8}$$

*with initial condition* $\phi(0,e,u) = e$.

**Proof:** We begin writing $W = Y + \sum u_j Y_j$. From Proposition 3.3, we see that $\exp(t(W,u)) = (\phi(t,e,u), t, u_1 t, \ldots, u_m t)$. Now Proposition 3.4 gives us

$(W,u)(\phi(t,e,u), t, u_1 t, \ldots, u_m t) =$

$$\begin{aligned} &= (W(\phi(t,e,u)) + \mathcal{X}(\phi(t,e,u)) + \sum u_j \mathcal{X}_j(\phi(t,e,u)), 1, u_1, \ldots, u_m) \\ &= \Big((\mathcal{X} + Y)(\phi(t,e,u)) + \sum u_j(\mathcal{X}_j + Y_j)(\phi(t,e,u)), 1, u_1, \ldots, u_m\Big). \end{aligned}$$

On the other hand,

$$(W, 1, \ldots, u_m)(\phi(t,e,u), t, u_1 t, \ldots, u_m t) = (d\phi(t,e,u)/dt, 1, \ldots, u_m).$$

So, in coordinates, it follows that

$$\begin{pmatrix} d\phi(t,e,u)/dt \\ 1 \\ u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} (\mathcal{X} + Y)(\phi(t,e,u)) + \sum u_j(\mathcal{X}_j + Y_j)(\phi(t,e,u)) \\ 1 \\ u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

Taking the projection on the first coordinate we see that the curve $\phi(t, e, u)$ satisfies the differential equation (3.8). Since $\phi(0, e, u) = e$, we conclude that $\phi(t, e, u)$ is the solution of the system at the identity. Finally, the description of $\phi(t, e, u)$ comes from Proposition 3.3. □

The previous theorem shows the solution of an Affine Control System at the identity. It is still possible to describe the solution at an arbitrary point $g \in G$. For this, we combine our result with [3, Theorem 4.1]. This theorem states the following: given the Affine Control System $\Sigma_A$ (3.8), we associate the following Bilinear Control System

$$\Sigma_B\colon \frac{dg}{dt} = \mathcal{X}(g) + \sum u_i \mathcal{X}_i(g). \tag{3.9}$$

Denoting by $\phi_A(t, \cdot, u)$, $\phi_B(t, \cdot, u)$ the solutions of $\Sigma_A$ and $\Sigma_B$, respectively, it follows that

$$\phi_A(t, g, u) = \phi_A(t, e, u)\phi_B(t, g, u)^{2},$$

for all $g \in G$ (As mentioned above, the proof can be found in [3, Theorem 4.1], page 8). Supposing the commutative condition $[\mathcal{X}_i, \mathcal{X}_j] = 0$, our next result gives an explicit description for $\phi_B(t, g, u)$.

**Corollary 3.6.** *The solution of the Dynamical System* (3.8) *at an arbitrary point $g \in G$ is given by*

$$\phi(t, g, u) = \phi(t, e, u)\rho(t, u_1 t, \dots, u_m t)(g)^{3}.$$

**Proof:** Consider a point $(g, r) \in G \times_\rho \mathbb{R}^{m+1}$, where $r = (r_0, \dots, r_m)$ is arbitrary. Let us denote by $\psi(t, (g, r), u)$ the solution of the system $\bar{\Sigma}$. Since $\bar{\Sigma}$ is an invariant system, it follows that $\psi(t, (g, r), u) = \psi(t, (e, 0), u) \cdot (g, r)$.

[2] The result is valid even for affine systems that do not satisfy the commutative condition (3.4).

[3] As we have said, this formula is the same of [3, Theorem 4.1]. We are redoing the calculations just to give an explicit description of $\phi_B(t, g, u)$.

On the other hand, we have that $\pi(\psi(t,(g,r),u)) = \phi(t,g,u)$. So

$$\begin{aligned}\phi(t,g,u) &= \pi\left(\psi(t,(e,0),u)(g,r)\right)\\ &= \pi((\phi(t,e,u),t,u_1t,\ldots,u_mt)\,(g,r_0,\ldots,r_m))\\ &= \pi(\phi(t,e,u)\rho(t,u_1t,\ldots,u_mt)(g),t+r_0,\ldots,u_mt+r_m)\\ &= \phi(t,e,u)\rho(t,u_1t,\ldots,u_mt)(g).\end{aligned}$$

□

**Example 3.1** (**Invariant Control Systems**)**.** An invariant control system is given by

$$\frac{dg}{dt} = Y(g) + \sum_{j=1}^{m} u_j Y_j(g),$$

where $Y$, $Y_1$, …, $Y_m$ are right invariant vector fields on $G$ and $u = (u_1,\ldots,u_m)$ is an admissible control. It is clear that it is a particular case of an affine system. In this case, the representation $\rho$ is the identity map. From Theorem 3.5 we compute

$$\phi(t,e,u) = \lim_{n\to\infty}\prod_{i=0}^{n-1}\exp\left(\frac{t}{n}\sum_{j=1}^{m}u_jY_j\right) = \exp\left(t\sum_{j=1}^{m}u_jY_j\right).$$

This result is already well-known in the literature. We present it here just to show the consistency of our solution.

**Example 3.2** (**Bilinear Control Systems**)**.** A bilinear control system is a control system defined by

$$\frac{dg}{dt} = \mathcal{X}(g) + \sum_{j=1}^{m} u_j \mathcal{X}_j(g),$$

where $\mathcal{X}$, $\mathcal{X}_1$, …, $\mathcal{X}_m$ are linear vector fields on $G$ and $u = (u_1,\ldots,u_m)$ is an admissible control. Since the identity is a singularity point[4] for this system, we are going to describe the solution at an arbitrary point $g \in G$. From Corollary 3.6 it follows immediately that

$$\phi(t,g,u) = \rho(t,u_1t,\ldots,u_mt)(g).$$

[4]We say that the point $g$ is a singularity of a control system on $G$ if $\phi(t,g,u) = g$, for all $t \in \mathbb{R}$ and all control function $u$.

In particular, if we consider a bilinear control system on $\mathbb{R}^n$ given by

$$\frac{dx}{dt} = \left( A + \sum_{i=1}^{m} u_i B_i \right) x,$$

where $A,\ B_i \in M(n \times n; \mathbb{R})$, it follows that the solution at a point $x$ is written as

$$\phi(t, x, u) = e^{tA} e^{u_1 t B_1} \cdots e^{u_m t B_m} x.$$

## 3.2 The Inner Derivation Case

In this section we study the solution curves and the controllability aspects of affine systems whose derivations associated to the linear vector fields of the system are inner. Our study extends a similar approach of the author of [12] for linear control system. We generalize two results from his work relating the controllability of the commutative affine control system to the controllability of an associated right invariant one (Theorems (3.10) and (3.12)). Our first step is to improve the description of the solutions.

Let $\mathcal{X}_0 = \mathcal{X},\ \mathcal{X}_1,\ \ldots,\ \mathcal{X}_m$ be linear vector fields on $G$. Suppose the associated derivations are all inner. That is, for each $i = 0,\ 1,\ \ldots,\ m$, there is a right invariant vector field $X_i \in \mathfrak{g}$ such that $\mathcal{D}_i = \mathrm{ad}(X_i)$, where $\mathcal{D}_i$ is the derivation associated to $\mathcal{X}_i$, respectively[5].

This fact implies that $\mathcal{X}_i = X_i + dIX_i$, where $dIX_i$ is the left invariant vector field induced by $I\colon G \to G$, $I(g) = g^{-1}$. Also, for each $i = 0,\ 1,\ \ldots,\ m$, we have that $\varphi_t^i(g) = \exp(tX_i) g \exp(-tX_i)$. Furthermore,

$$[\mathcal{X}_i, \mathcal{X}_j] = 0 \Leftrightarrow [X_i, X_j] = 0.$$

**Theorem 3.7.** Let $\Sigma_A$ (3.1) be an affine control system. Suppose that the derivation $\mathcal{D}_i$ associated to the linear vector field $\mathcal{X}_i$ is inner, for all $i = 0,\ 1,\ \ldots,\ m$. Then, the solution

[5] We recall again that are included in this case all semi simple Lie groups since every derivation defined on their Lie algebras are inner (see for instance [21]).

curve of $\Sigma_A$ at the identity is

$$\phi(t,e,u) = \exp\left(tX + tY + \sum_{i=1}^{m} u_i t(X_i + Y_i)\right)\exp\left(-t\sum_{i=0}^{m} u_i X_i\right). \tag{3.10}$$

**Proof:** We first write $W = Y + \sum u_i Y_i$. Consider $u_0 = 1$. Since $\varphi_t^i(g) = \exp(tX_i)g\exp(-tX_i)$ for each $\varphi_i$, it follows that

$$\begin{aligned}
\phi(t,e,u) &= \lim_{n\to\infty}\prod_{i=0}^{n-1}\rho(it/n, iu_1t/n,\ldots,iu_mt/n)\exp\left(\frac{t}{n}W\right)\\
&= \lim_{n\to\infty}\prod_{i=0}^{n-1}\left(\prod_{k=0}^{m}\exp\left(\frac{iu_kt}{n}X_k\right)\exp\left(\frac{t}{n}W\right)\prod_{k=0}^{m}\exp\left(-\frac{iu_{m-k}t}{n}X_{m-k}\right)\right)\\
&= \lim_{n\to\infty}\prod_{i=0}^{n-1}\left(\exp\left(\sum_{k=0}^{m}\frac{iu_kt}{n}X_k\right)\exp\left(\frac{t}{n}W\right)\exp\left(-\sum_{k=0}^{m}\frac{iu_kt}{n}X_k\right)\right),
\end{aligned}$$

where we use the fact that $[X_i, X_j] = 0$ for $i, j = 0, 1, \ldots, n$. Computing the product we get

$$\phi(t,e,u) =$$

$$\lim_{n\to\infty}\left(\exp\left(\frac{t}{n}W\right)\exp\left(\sum_{k=0}^{m}\frac{u_kt}{n}X_k\right)\right)^{n-1}\exp\left(\frac{t}{n}W\right)\exp\left(\sum_{k=0}^{m}\frac{(1-n)u_kt}{n}X_k\right).$$

Inserting

$$\exp\left(\frac{t}{n}W\right)\exp\left(\sum_{k=0}^{m}\frac{u_kt}{n}X_k\right)\exp\left(-\sum_{k=0}^{m}\frac{u_kt}{n}X_k\right)\exp\left(-\frac{t}{n}W\right)$$

in right side above yields

$$\phi(t,e,u) = \lim_{n\to\infty}\left(\exp\left(\frac{t}{n}W\right)\prod_{k=0}^{m}\exp\left(\frac{u_kt}{n}X_k\right)\right)^{n}\exp\left(-t\sum_{i=0}^{m}u_iX_i\right).$$

Finally, applying the Lie Product Formula we obtain

$$\phi(t,e,u) = \exp\left(tX + tY + \sum_{i=1}^{m} u_i t(X_i + Y_i)\right)\exp\left(-t\sum_{i=0}^{m} u_i X_i\right).$$

□

If we look closely, we realize that the first factor in the above formula is the solution of an invariant system. Thus, it is natural to associate to $\Sigma_A$ the following control system on $G$:

$$\Sigma_I\colon \frac{dg}{dt} = (X+Y)(g) + \sum_{i=1}^{m} u_i(X_i+Y_i)(g),$$

where $X_i$ is such that $\mathcal{X}_i = X_i + dIX_i$ for $i = 0, \dots, m$. Although the last result was enunciated for constant control functions, it suggests the following global result, which establishes a relation between the affine control system $\Sigma_A$ and the right invariant control system $\Sigma_I$.

**Proposition 3.8.** *Let us denote $t \to \phi_A(t)$ and $t \to \phi_I(t)$ the solutions of the Affine and of the Invariant Control Systems at the identity, respectively. Then*

$$\phi_A(t) = \phi_I(t)\exp\left(-t\Sigma u_i X_i\right),$$

*for all control function $u$.*

**Proof:** We first simplify the notation by writing $\alpha(t) = \exp\left(t\Sigma u_i X_i\right)$. We intend to prove that $\phi_A(t)$ is the solution of $\Sigma_A$ if, and only if, $\phi_I(t) = \phi_A(t)\alpha(t)$ is the solution of $\Sigma_I$. Suppose initially that $\phi_A$ is the solution for $\Sigma_A$. Differentiating $\phi_A(t)\alpha(t)$ yields

$$\begin{aligned}\frac{d}{dt}\left(\phi_A(t)\alpha(t)\right) &= dR_{\alpha(t)}\frac{d}{dt}\phi_A(t) + dL_{\phi_A(t)}\frac{d}{dt}\alpha(t)\\ &= dR_{\alpha(t)}\left((\mathcal{X}+Y)\phi_A(t) + \sum u_i(\mathcal{X}_i+Y)\phi_A(t)\right) + dL_{\phi_A(t)}\frac{d}{dt}\alpha(t).\end{aligned}$$

Now, writing $\mathcal{X}_i = X_i + dIX_i$ and using the right invariance property we obtain

$$\frac{d}{dt}(\phi_A(t)\alpha(t)) = \Sigma_I(\phi_I(t)) + dR_{\alpha(t)}\left(dIX + \sum u_i dIX_i\right)\phi_A(t) + dL_{\phi_A(t)}\frac{d}{dt}\alpha(t).$$

The result follows since

$$dR_{\alpha(t)}\left(dIX + \sum u_i dIX_i\right)\phi_A(t) + dL_{\phi_A(t)}\frac{d}{dt}\alpha(t) = 0.$$

The converse is proved similarly. □

In the following, we write $\mathcal{S}$ and $\mathcal{A}$ to denote the reachable sets of the invariant and affine systems, respectively. In particular, for any $t > 0$ $\mathcal{S}_t$ is the reachable set at time $t$. In the next results we intend to relate the controllability of the affine system and the associated right invariant one. Before that we need a lemma. We abbreviate the notation writing $\rho(t, u)$ instead of $\rho(t, u_1 t, \ldots, u_m t)$, for $t > 0$ and $u = (u_1, \ldots, u_m)$.

**Lemma 3.9.** For each control function $u$ and $t,\ s > 0$ we have $\phi(t, e, u)\rho(t, u)(\mathcal{A}_s) \subset \mathcal{A}_{t+s}$.

**Proof:** Choose an arbitrary $\phi(s, e, v) \in \mathcal{A}_s$. Define $u' = v \wedge_s u$, the $s$ - concatenation of the control functions $v$ and $u$. Then

$$\begin{aligned} \phi(t, e, u)\rho(t, u)(\phi(s, e, v)) &= \phi(t, e, \theta_s u')\rho(t, \theta_s u')(\phi(s, e, u')) \\ &= \phi(t, \phi(s, e, u'), u') = \phi(t + s, e, u'). \end{aligned}$$

□

At last, observe that for any control function $u$ and $t > 0$, it is true that

$$\rho(t, u)\left(\exp\left(t\left(X_0 + \sum u_i X_i\right)\right)\right) = \exp\left(t\left(X_0 + \sum u_i X_i\right)\right).$$

This property follows from the definition of $\rho$ and the fact that $\phi_t(g) = \exp(tX)g\exp(-tX)$.

Now we state a result relating the controllability of systems $\Sigma_A$ and $\Sigma_I$. It is a generalization of [12, Theorem 1], page 13.

**Theorem 3.10.** *Suppose that the right invariant system $\Sigma_I$ is controllable. The following assertions are equivalent:*

*(i) For all control function $u$ and all $t \in \mathbb{R}$, $\exp\left(t\left(X_0 + \sum u_i X_i\right)\right) \in \mathcal{A}$;*

*(ii) $\Sigma_A$ is controllable.*

**Proof:** The assertion *(ii)⇒(i)* is immediate. So we prove *(i)⇒(ii)*.

We prove first that $\Sigma_A$ is controllable from the identity $e$. For simplicity of notation, write $\exp(tX_u) = \exp\left(t\left(X_0 + \sum u_i X_i\right)\right)$. Given $g \in G$, there are a piecewise constant control $u$ and a time $t > 0$ such that $g = \phi_I(t, e, u) \in \mathcal{S}_t$. This is equivalent to $g \exp(-tX_u) \in \mathcal{A}_t$. By hypothesis, $\exp(tX_u) \in \mathcal{A}_s$ for some $s > 0$. Thus,

$$g = g \exp(-tX_u)\rho(t,u)(\exp(tX_u)) \in \mathcal{A}_t \rho(t,u)(\mathcal{A}_s) = \mathcal{A}_{t+s} \subset \mathcal{A}.$$

Now, we prove that $\Sigma_A$ is controllable to $e$. Fix $g \in G$. By assumption, there are $t > 0$ and a control $u$ such that $g^{-1} = \phi_I(t, e, u) \in \mathcal{S}_t$. This is equivalent to $g^{-1} \exp(-tX_u) \in \mathcal{A}_t$. On one hand, we have that

$$\begin{aligned} \exp(-tX_u) &= g^{-1} \exp(-tX_u) \exp(tX_u) g \exp(-tX_u) = g^{-1} \exp(-tX_u)\rho(t,u)(g) \\ &= \phi_A(t, e, u)\rho(t,u)(g) = \phi_A(t, g, u) \in \mathcal{A}_t(g), \end{aligned}$$

where we use Corollary 3.6 at the last equality. On the other hand, we have $\exp(tX_u) \in \mathcal{A}_s$ for some $s > 0$. It means that there exists a piecewise constant control $u'$ such that $\exp(tX_u) = \phi_A(t, e, u')$. Let $X_{u'} = X_0 + \sum u'_i X_i$. Since $[X_u, X_{u'}] = 0$, it follows that

$$\begin{aligned} e &= \exp(tX_u) \exp(sX_{u'}) \exp(-tX_u)(\exp(-sX_{u'})) \\ &= \exp(tX_u)\rho(s,u')(\exp(-tX_u)) \\ &= \phi_A(s, e, u')\rho(s,u')(\phi_A(t, g, u)) = \phi_A(s, e, u'')\rho(s,u'')(\phi_A(t, g, u'')), \end{aligned}$$

where $u''$ is the concatenation of $u$ and $u'$ (just as in the proof of Lemma 3.9). It follows that

$$e = \phi_A(t, \phi_A(s, g, u''), u'') = \phi_A(t + s, g, u'') \in \mathcal{A}_{t+s}(g) \subset \mathcal{A}(g).$$

We thus conclude that $e \in \mathcal{A}(g)$, and the proof is complete. □

**Corollary 3.11.** *Under the hypothesis of the previous theorem, if* $e \in \operatorname{int}(\mathcal{A}_T)$ *for some* $T > 0$*, then* $\Sigma_A$ *is controllable.*

**Proof:** Given a control function $u$, we write $X_u = X_0 + \sum u_i X_i$. Let $S_u = \{t \in \mathbb{R} \colon \exp(tX_u) \in \mathcal{A}\}$. An analysis similar to that in the proof of the previous theorem shows that $S_u$ is a semigroup. By hypothesis, $\mathcal{A}_T$ is a neighborhood of $e$ for some $T > 0$.

It implies that $0 \in S_u$ since $\exp(0X) = e \in \mathcal{A}$. For each admissible control $u$ the curve $\exp(tX_u)$ is continuous. Then $\exp(sX_u) \in \mathcal{A}_t$ for $s \in (-\epsilon, \epsilon)$. In particular, $(-\epsilon, \epsilon) \subset S_u$. Being $S_u$ a semigroup, it follows that $S_u = \mathbb{R}$. It means that for all control function $u$ and all $t \in \mathbb{R}$, $\exp\left(t\left(X_0 + \sum u_i X_i\right)\right) \in \mathcal{A}$. According to the above theorem, $\Sigma_A$ is controllable. □

To finish this section we give a result that generalizes for affine control systems the Theorem 2 in [12]. First, we need to recall that a semigroup $S \subset G$ is said to be left reversible (resp. right reversible) if $SS^{-1} = G$ (resp. $S^{-1}S = G$). It is known that if $G$ is semi simple with finite center, the unique subsemigroup of $G$ with nonempty interior which is left or right reversible is $G$ itself (see for instance [23]).

**Theorem 3.12.** *Let $G$ be a semi simple Lie group with finite center. Suppose that $\Sigma_I$ satisfies the rank condition and $\Sigma_A$ is controllable. The following assertions are equivalent:*

*(i) For all $u \in \mathcal{U}$ and all $t \in \mathbb{R}$, $\exp\left(t\left(X_0 + \sum u_i X_i\right)\right) \in \mathcal{S}$;*

*(ii) $\Sigma_I$ is controllable.*

**Proof:** We first observe that it is direct that *(ii)* implies *(i)*. Let us prove the converse. We begin by recalling that the reachable set $\mathcal{S}$ of $\Sigma_I$ is a semigroup. Now, the rank condition assures that the interior of $\mathcal{S}$ is non-empty. It is sufficient to prove that $\mathcal{S}$ is left reversible. Fix $g \in G$. There are $t > 0$ and $u \in \mathcal{U}$ such that $g = \phi_I(t, e, u)\exp(-tX_u)$. By assumption, $\exp(-tX_u) \in \mathcal{S}^{-1}$. Then $g \in \mathcal{S}\mathcal{S}^{-1}$. Since $g \in G$ was chosen arbitrarily, we conclude $G \subset \mathcal{S}\mathcal{S}^{-1}$, and the result follows. □

## 3.3 Conjugation of Affine Systems

This last section is intended to study the concept of conjugation of affine systems, which is the structure-preserving tool or the concept responsible for identifying control systems that, although may look different, have indistinguishable behaviors with respect to control theory. Despite there are other notions of conjugation, here we consider conjugations of affine systems by group homomorphisms.

Let $G$ and $H$ be connected Lie groups. Consider the following affine systems

$$\frac{dg}{dt} = (\mathcal{X} + Z)(g) + \sum_{i=1}^{m} u_i(\mathcal{X}_i + Z_i)(g) \tag{3.11}$$

$$\frac{dh}{dt} = (\mathcal{Y} + W)(h) + \sum_{i=1}^{m} u_i(\mathcal{Y}_i + W_i)(h) \tag{3.12}$$

defined on $G$ and $H$, respectively. Affine systems are said to be semiconjugate if there exist a homomorphism of Lie groups $F\colon G \to H$ such that

$$F(\phi(t, g, u)) = \theta(t, F(g), u),$$

where $\phi(t, g, u)$, $\theta(t, g, u)$ are the respective solutions of the systems. Moreover, if $F$ is a Lie group isomorphism, the two systems have exactly the same (control related) properties. That is, one is (locally) controllable if, and only if, the other is, etc. In such cases, they are said to be conjugate.

Let us denote by $\varphi_t^i$, $\psi_t^i$ the flows and by $D_{\mathcal{X}_i}$, $D_{\mathcal{Y}_i}$ the derivations associated to the linear vector fields $\mathcal{X}_i$ and $\mathcal{Y}_i$, respectively, $i = 0, 1, \dots, m$[6]. We give equivalent conditions for two affine systems to be semiconjugate.

**Proposition 3.13.** *Let $F\colon G \to H$ be a homomorphism of Lie groups. The following conditions are equivalent:*

*1) $F \circ \varphi_t^i = \psi_t^i \circ F$;*

*2) $dF_{\varphi_t^i(g)}\mathcal{X}_i(g) = \mathcal{Y}_i(F(g))$ for all $g \in G$;*

*3) $dF_e(e^{tD_{\mathcal{X}_i}}Z) = e^{tD_{\mathcal{Y}_i}}dF_eZ$, for all $Z \in \mathfrak{g}$.*

**Proof:** To deduce 2) from 1) differentiate formula in 1) with respect to $t$ to obtain $dF_{\varphi_t^i}\dfrac{d\varphi_t^i}{dt}(g) = \dfrac{d}{dt}(\psi_t^i \circ F(g))$. We thus get

$$dF_{\varphi_t^i(g)}\mathcal{X}_i(g) = \mathcal{Y}_i(F(g)).$$

[6] Recall that, throughout this work, $\mathcal{X}_0 = \mathcal{X}$ and $Y_0 = Y$.

Conversely, to deduce 1) from 2), observe that it is a direct consequence of uniqueness of solution of differential equations.

Suppose now that 1) is true. Then $dF_e \circ (d\varphi_t^i)_e = d(\psi_t)_e \circ dF_e$. Since $(d\varphi_t^i)_e = e^{tD\mathcal{X}_i}$ and $(d\psi_t^i)_e = e^{tD\mathcal{Y}_i}$, it follows for all $Z \in \mathfrak{g}$ that

$$dF_e(e^{tD\mathcal{X}_i}Z) = dF_e \circ (d\varphi_t^i)_e(Z) = (d\psi_t^i)_e \circ (dF_e)(Z) = e^{t\mathcal{D}\mathcal{Y}_i}dF_e(Z).$$

On the converse, to deduce 1) to 3) observe that it is a direct consequence of $d(F \circ \varphi_t^i)_e = d(\psi_t \circ F)_e$, since $G$ is connected. □

**Theorem 3.14.** *Let $F\colon G \to H$ be a homomorphism of Lie groups. The following conditions are equivalent:*

*1) $F(\phi(t, g, u)) = \theta(t, F(g), u)$ for all $g \in G$.*

*2) $F \circ \varphi_t^i = \psi_t^i \circ F$ and $dF_e Z_j(e) = W_j(e)$, for all $i, j = 0, 1, \ldots, m$.*

**Proof:** We first suppose that $F(\phi(t, g, u)) = \theta(t, F(g), u)$ for all $g \in G$. In particular, $F(\phi(t, e, u)) = \theta(t, e, u)$. For abbreviation, we write $\phi_t$ and $\theta_t$ instead of $\phi(t, e, u)$ and $\theta(t, e, u)$, respectively. Differentiating 1) with respect to $t$ yields

$$dF_{\phi_t}\left((\mathcal{X} + Z)\phi_t + \sum_{j=1}^{m} u_j(\mathcal{X}_j + Z_j)\phi_t\right) = (\mathcal{Y} + W)\theta_t + \sum_{j=1}^{m} u_j(\mathcal{Y}_j + W_j)\theta_t.$$

Taking $t = 0$ it follows that

$$dF_e\left(Z(e) + \sum_{j=1}^{m} u_j(0)Z_j(e)\right) = W(e) + \sum_{j=1}^{m} u_j(0)W_j(e).$$

The above equality holds for all control $u(t) = (u_1(t), \ldots, u_m(t))$. If $u \equiv 0$, then $dF_e Z(e) = W(e)$. If $u \equiv (0, \ldots, 1, \ldots, 0)$, where the 1 stands in the $j$ - th position, then $dF_e Z_j(e) = W_j(e)$, for $j = 1, \ldots, m$. It gives

$$dF_g\left(\mathcal{X}(g) + \sum_{j=1}^{m} u_j\mathcal{X}_j(g)\right) = \mathcal{Y}(F(g)) + \sum_{j=1}^{m} u_j\mathcal{Y}_j(F(g)).$$

In the same manner we can see that $dF_{\phi_t}\mathcal{X}_i(g) = \mathcal{Y}_i(F(g))$. From the above proposition it follows that $F \circ \varphi_t^i = \psi_t^i \circ F$, for $i = 0, \ldots, m$.

Conversely, let us denote by $\rho\colon \mathbb{R}^{m+1} \to Aut(G)$ and $\varrho\colon \mathbb{R}^{m+1} \to Aut(H)$ the representations associated to the Affine Systems (3.11) and (3.12), respectively. Assuming that condition 2) is true we compute

$$\begin{aligned} F(\phi(t,e,u)) &= F\left(\lim_{n\to\infty}\prod_{i=0}^{n-1}\rho(it/n, iu_1t/n, \dots, iu_mt/n)\exp\left(\frac{t}{n}\sum_{j=0}^{m}u_jZ_j\right)\right) \\ &= \lim_{n\to\infty}\prod_{i=0}^{n-1}\varrho(it/n, iu_1t/n, \dots, iu_mt/n)\exp\left(\frac{t}{n}\sum_{j=0}^{m}u_jdF_eZ_j\right), \end{aligned}$$

where we use that $F\circ\rho = \varrho\circ F$ and that $F\circ\exp = \exp\circ dF_e$. This gives $F(\phi(t,e,u)) = \theta(t,e,u)$. From this last equality and Corollary 3.6 we see that

$$\begin{aligned} F(\phi(t,g,u)) &= F(\phi(t,e,u)\rho(t,u_1t,\dots,u_mt)(g)) \\ &= F(\phi(t,e,u))F(\rho(t,u_1t,\dots,u_mt)(g)) \\ &= \theta(t,e,u)\varrho(t,u_1t,\dots,u_mt)(F(g)) = \theta(t,F(g),u). \end{aligned}$$

□

**Corollary 3.15.** *Two affine systems defined on $G$ and $H$ are semiconjugate if, and only if, they are semiconjugate at the identity $e\in G$.*

**Proof:** See the end of the proof of the previous theorem. □

Lastly, we characterize conjugation of affine system in terms of derivations.

**Corollary 3.16.** *Let $F\colon G\to H$ be a homomorphism of Lie groups. A necessary and sufficient condition for the Affine Control Systems (3.11) and (3.12) to be semiconjugate is $dF_e\left(e^{D_t}Z_j\right) = e^{D'_t}W_j$, for all $j = 0, 1, \dots, m$, $t\in\mathbb{R}$, where $D_t = \sum\frac{iu_kt}{n}D_{\mathcal{X}_k}$ and $D'_t = \sum\frac{iu_kt}{n}D_{\mathcal{Y}_k}$.*

**Proof:** It is sufficient to show the necessary condition. Write the solution of the Affine System (3.11) as

$$\phi(t,e,u) = \lim_{n\to\infty}\prod_{i=0}^{n-1}\exp\left(t/n\cdot\sum u_je^{D_t}Z_j\right).$$

Assuming that the systems are semiconjugate and computing $F(\phi(t,e,u)$, we have that the formula

$$F(\phi(t,e,u)) = \lim_{n\to\infty} \prod_{i=0}^{n-1} \exp\left(t/n \cdot \sum u_j dF_e\left(e^{D_t} Z_j\right)\right),$$

is the solution of the System (3.12). Writing the solution of this last system explicitly, the result follows by comparison. □

**Example 3.3.** Consider the homomorphism $\det\colon Gl(n;\mathbb{R})^+ \to \mathbb{R}$ and the linear system (2.4) defined in Example 2.2. Let us construct a control system on $\mathbb{R}$ semiconjugate to it. We first need to find a linear vector field $\mathcal{Y}$ and invariant vector fields $b_1, \ldots, b_m$ on $\mathbb{R}$ satisfying conditions

1. $\det\left(e^{tA} \cdot g \cdot e^{-tA}\right) = \psi_t(\det(g))$, for all $g \in Gl(n;\mathbb{R})^+$, where $\psi_t$ is the flow of $\mathcal{Y}$;
2. $d(\det)_I(B_j) = tr(B_j) = b_j$.

Condition 2) gives the invariant vector fields $b_j$, $j = 1, \ldots, m$. Also, condition 1) implies that $\psi_t(\det(g)) = \det(g)$ for all $g$. This clearly forces $\mathcal{Y} = 0$. We thus conclude that the linear system (2.4) on $Gl(n;\mathbb{R})^+$ is semiconjugate to the following invariant system on $\mathbb{R}$

$$\frac{dx}{dt} = \sum_{j=1}^{m} tr(B_j).$$

Chapter 4

# Geometric Properties and Stable Sets

This chapter is an initial study of some dynamical aspects of control systems on Lie groups. Our initial approach is to study the metric properties of the most frequent control systems in control theory of Lie groups: linear, bilinear and affine. For us to refer to a particular control system on a Lie group $G$, we establish the following notations, which are used throughout this chapter:

$\Sigma_I$: Right Invariant Control System:

$$\frac{dg}{dt} = Y(g) + \sum u_i Y_i(g);$$

$\Sigma_L$: Linear Control System:

$$\frac{dg}{dt} = \mathcal{X}(g) + \sum u_i Y_i(g);$$

$\Sigma_B$: Bilinear Control System:

$$\frac{dg}{dt} = \mathcal{X}(g) + \sum u_i \mathcal{X}_i(g);$$

$\Sigma_A$: Affine Control System:

$$\frac{dg}{dt} = (\mathcal{X} + Y)(g) + \sum u_i(\mathcal{X}_i + Y_i)(g).$$

In the above notation, as well as throughout the whole work, $\mathcal{X}$, $\mathcal{X}_1$, $\dots$, $\mathcal{X}_m$ are linear while $Y$, $Y_1$, $\dots$, $Y_m$ are right invariant vector fields. In discussions involving simultaneously more than one of the above systems, we will use the notations

$$\phi_I(t,x,u), \quad \phi_L(t,x,u), \quad \phi_B(t,x,u) \quad \text{and} \quad \phi_A(t,x,u).$$

These notations stand for the solutions of the Invariant, Linear, Bilinear and Affine Control Systems, respectively. Otherwise, if in the statement of a result only one of the systems is involved, we continue to write its solutions simply by $\phi(t,x,u)$, without danger of confusion.

Finally, we write $\varrho\colon G \times G \to \mathbb{R}$ to denote either a right or left riemannian distance on $G$, according to the convenience. Given $x \in G$ and $\epsilon > 0$, $B(x;\epsilon)$ is the set of all points $y \in G$ such that $\varrho(x,y) < \epsilon$.

## 4.1 Linear Control Systems

In this section, we present some specific results concerning linear control systems. For the results to come, we consider a left invariant riemannian distance $\varrho$ on $G$. We begin showing that, for each fixed control function $u$, the distance between the points of two distinct trajectories depends only on the linear flow of the system, not on the control function associated to the trajectories.

**Proposition 4.1.** *Let $\varrho$ be a left invariant riemannian distance on $G$. For all $g, h \in G$ it follows that*

$$\varrho(\phi(t,g,u), \phi(t,h,u)) = \varrho(\varphi_t(g), \varphi_t(h)).$$

**Proof:** The result follows immediately from the property $\phi(t,g,u) = \phi(t,e,u)\varphi_t(g)$ and the left invariance of the metric. □

Note that the above result does not depend on the control function $u$. Thus, it is valid for all trajectories of the system.

**Corollary 4.2.** *For all $g \in G$, $\varrho(\phi(t,g,u), \phi(t,e,u)) = \varrho(\varphi_t(g), e)$.*

For the next result, we assume that the linear vector field is hyperbolic. That is, the derivation associated to $\mathcal{X}$ has no eigenvalues with zero real part. As pointed in [25, Remark 3, p. 4], this assumption already implies that $G$ must be a nilpotent Lie group. The next result characterizes the trajectories of the system that starts at the identity as an attractor or a repeller for the other trajectories. In order to state the result, we present the following notation: given a linear vector field $\mathcal{X}$, we associate the following $\mathcal{D}$ - invariant subalgebras

$$\mathfrak{g}^+ = \bigoplus_{\alpha;\,\mathrm{Re}(\alpha)>0} \mathfrak{g}_\alpha, \qquad \mathfrak{g}^0 = \bigoplus_{\alpha;\,\mathrm{Re}(\alpha)=0} \mathfrak{g}_\alpha, \qquad \text{and} \qquad \mathfrak{g}^- = \bigoplus_{\alpha;\,\mathrm{Re}(\alpha)<0} \mathfrak{g}_\alpha,$$

where $\alpha$ is an eigenvalue of $\mathcal{D}$. We consider $G^+$, $G^-$ and $G^0$ the connected $\varphi$ - invariant subgroups of $G$ with Lie algebras $\mathfrak{g}^+$, $\mathfrak{g}^-$ and $\mathfrak{g}^0$, respectively. For important properties about these subgroups and subalgebras, see [26].

**Theorem 4.3.** *Let $\mathcal{X}$ be an hyperbolic linear vector field. The following assertions hold:*

1) *$g \in G^-$ if and only if $\lim_{t\to\infty} \varrho(\phi(t,g,u), \phi(t,e,u)) = 0$;*

2) *$g \in G^+$ if and only if $\lim_{t\to-\infty} \varrho(\phi(t,g,u), \phi(t,e,u)) = 0$*

**Proof:** According to [25, Theorem 2.5, p. 4], under the above hypothesis, a point $g$ belongs to $G^-$ if and only if $\lim_{t\to\infty} \varrho(\varphi_t(g), e) = 0$. Analogously, $g \in G^+$ if and only if $\lim_{t\to-\infty} \varrho(\varphi_t(g), e) = 0$. From Corollary 4.2, we have that $\varrho(\phi(t,g,u), \phi(t,e,u)) = \varrho(\phi_t(g), e)$. The result follows. □

As Theorem 4.3 shows, it is interesting that trajectories departing from the identity work as an attractor for the trajectories starting at points $g \in G^-$ (as long as we consider trajectories with the same control function). Similarly, for solutions of points $g \in G^+$, the solutions at the identity has a repeller behavior.

Considering the results established so far, in addition to the facts commented above, we can obtain other interesting conclusions about the dynamical behavior of the trajectories of a linear system in the context of control sets, which are defined up next.

**Definition 4.4.** *A set $\mathcal{C} \subset M$ is called a control set of a control system if it is:*

1) *controlled invariant: for all $x \in \mathcal{C}$ there is a control function $u$ such that $\phi(t,x,u) \in \mathcal{C}$, for all $t \geq 0$.*

2) *approximate controllable: for all $x \in \mathcal{C}$, holds the inclusion $\mathcal{C} \subset \overline{\mathcal{A}(x)}$.*

*3) maximal with properties 1) and 2).*

Condition 1) is included only to exclude trivial cases since any one-point set satisfies condition 2). It is satisfied if $\mathcal{C}$ has non-void interior and satisfies condition 2). In its turn, this last property means that we can get close enough to a point in $\mathcal{C}$ from any other of its points. Finally, condition 3) is for simplicity. It also implies that any two distinct control sets are disjoint. A control set $\mathcal{C}$ is said to be invariant if $\overline{\mathcal{C}} = \overline{\mathcal{A}(x)}$.

**Definition 4.5.** *A set $L \subset M$ is positively-invariant if $\mathcal{A}(x) \subset L$ for all $x \in L$.*

**Proposition 4.6.** *Let $\mathcal{C} \subset M$ be an invariant control set and assume that the system satisfies the accessibility rank condition in the closure of $\mathcal{C}$. Then:*

*1)* $\text{int}(\mathcal{C})$ *is non-void.*

*2)* $\overline{\text{int}(\mathcal{C})} = \mathcal{C}$ *and $C$ is connected.*

*3)* $\text{int}(\mathcal{C})$ *and $\mathcal{C}$ are positively-invariant.*

*4)* $\text{int}(\mathcal{C}) \subset \mathcal{A}(x)$ *for all $x \in \mathcal{C}$, satisfying the equality if $x \in \text{int}(C)$.*

*5) if the system is full-rank, there are at most countably many invariant control sets in $M$.*

**Proof:** See [10, Lemma 3.2.7], page 55. □

Note that the statement 4) in the above proposition assures that controllability occurs in the interior of $\mathcal{C}$.

The next result shows that the control set $\mathcal{C}$ can work as an attractor or a repeller for the trajectories of the system.

**Corollary 4.7.** *Consider a linear control system defined on a connected Lie group $G$. Suppose that the derivation associated to the linear vector field has only eigenvalues with negative real part and that the control set $\mathcal{C}$ that contains the identity is positively-invariant. Then for all control function $u$ and all $g \in G$,*

$$\lim_{t\to\infty} \varrho(\mathcal{C}, \phi(t, x, u)) = 0.$$

*In other words, all trajectories of the system converge to the control set $\mathcal{C}$.*

**Proof:** It is an immediate consequence of Corollary 4.3 since for all control functions $u$, we have $\phi(t, e, u) \in \mathcal{C}$. □

Obviously, we have an analogous result if we suppose that the eigenvalues of the derivations have only positive real parts. But in this case, the conclusion would be

$$\lim_{t \to -\infty} \varrho(\mathcal{C}, \phi(t, x, u)) = 0.$$

## 4.2 Bilinear Control Systems

For bilinear control systems, even under the assumption that all linear vector fields are hyperbolic, it is harder to predict the behavior of all trajectories of the system. However, for points that belong to a certain subgroup and for control functions that take values on a convex cone of $\mathbb{R}^m$, we can show that all trajectories converge to the group identity.

Consider the Bilinear Control System

$$\Sigma_B : \frac{dg}{dt} = \mathcal{X}(g) + \sum u_i \mathcal{X}_i(g)$$

defined on $G$. We denote $G_i^+$ and $G_i^-$ the unstable and stable subgroups associated to the linear vector field $\mathcal{X}_i$, $i = 0, 1, \ldots, m$. We state the following result:

**Theorem 4.8.** *Suppose that each linear vector field $\mathcal{X}_i$ is hyperbolic, $i = 0, 1, \ldots, m$. Let $u(t) = (u_1(t), \ldots, u_m(t))$ be a control function such that $u_i(t) \geq 0$. It follows that*

*1) If $g \in G^- = G_0^- \cap \cdots \cap G_m^-$, then*

$$\lim_{t \to \infty} \phi(t, g, u) = e.$$

*2) If $g \in G^+ = G_0^+ \cap \cdots \cap G_m^+$, then*

$$\lim_{t \to -\infty} \phi(t, g, u) = e.$$

**Proof:** We prove just item 1) since item 2) is analogous.

Let us suppose initially that the control function $u = (u_1, \ldots, u_m)$ is constant. For short, we write simply $\phi_t(g) = \phi(t, g, u)$. Before starting the proof, we give an extended version for bilinear systems of some considerations presented in [3, Remark 3], p. 4.
For a left invariant riemannian distance $\varrho$ on $G$, we have

$$\varrho(\phi_t(g), \phi_t(h)) \leq \|d(\phi_t)_e\| \varrho(g, h), \qquad \text{for all} \qquad g, h \in G, \;\; t \geq 0.$$

Now, let

$$\mathcal{D} = \mathcal{D}_0 + \sum u_i D_i,$$

$$\mathfrak{g}^+ = \mathfrak{g}_0^+ \cap \cdots \cap \mathfrak{g}_m^+,$$

$$\mathfrak{g}^- = \mathfrak{g}_0^- \cap \cdots \cap \mathfrak{g}_m^-.$$

Consider $\varrho^+$, $\varrho^-$ the riemannian distances induced by $\varrho$ on $G^+$ and $G^-$, respectively. Since $d(\phi_t^-)_e = e^{t\mathcal{D}|_{\mathfrak{g}^-}}$ has only eigenvalues with negative real part, there are constants $c$, $\mu > 0$ such that

$$\varrho^-(\phi_t^-(g), \phi_t^-(h)) \leq c^{-1} e^{-\mu t} \varrho^-(g, h), \qquad \text{for all} \qquad g, h \in G^-, \;\; t \geq 0. \tag{4.1}$$

Analogously, we have

$$\varrho^+(\phi_t^+(g), \phi_t^+(h)) \geq c e^{\mu t} \varrho^+(g, h), \qquad \text{for all} \qquad g, h \in G^+, \;\; t \geq 0. \tag{4.2}$$

Now suppose 1). Let $g \in G^- = G_0^- \cap \cdots \cap G_m^-$. Since $\phi(t, g, u) = \phi^-(t, g, u) \in G^-$ it follows from equation (4.1) that

$$\varrho^-(\phi(t, g, u), e) = \varrho^-(\phi^-(t, g, u), \phi^-(t, e, u)) \leq c^{-1} e^{-\mu t} \varrho^-(g, e).$$

Taking $t \to \infty$, we have that $\varrho^{-}(\phi(t,g,u),e) \to 0$, that is $\lim_{t\to\infty} \phi(g,u,t) = e$.

Finally, for an arbitrary piecewise constant control function, the trajectory $\phi(t,g,u)$ is a concatenation of trajectories with constant control functions. The result follows from the case proved above. □

**Example 4.1.** Define the bilinear system

$$\frac{dg}{dt} = \mathcal{X}(g) + u\mathcal{Y}(g)$$

on the Heisenberg group, identified with $\mathbb{R}^3$ just as in Example 2.1. The linear vector fields were chosen in such a way that the derivations associated to them are

$$\mathcal{D}_{\mathcal{X}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{D}_{\mathcal{Y}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

A direct calculation gives $G^{-} = \mathbb{R} \times 0 \times 0$. We use Example 3.2 to write the solution of the system for an arbitrary control function $u \geq 0$ and $g = (x,0,0) \in G^{-}$. The result obtained is

$$\phi(t,g,u) = e^{t\mathcal{D}_{\mathcal{X}}} \cdot e^{ut\mathcal{D}_{\mathcal{Y}}} \cdot (x,0,0) = \left(e^{(-1-u)t}x, 0, 0\right).$$

Clearly, we have $\phi(t,g,u) \to (0,0,0)$ as $t \to \infty$.

## 4.3 Affine Control Systems

The last particular kind of control system considered is the affine. Compared to the previous systems, this is the one whose behavior of the solutions is the most complicated. This is due to the fact that affine systems generalize all the previous ones and contain a greater amount of information in their definition. However, it is possible to give analogous results about their solutions in some cases. We begin presenting a version of Proposition 4.1.

**Proposition 4.9.** *Let $\varrho$ be a left invariant riemannian distance on $G$. For all $g, h \in G$ we have*

$$\varrho(\phi_A(t,g,u), \phi_A(t,h,u)) = \varrho(\phi_B(t,g,u), \phi_B(t,h,u)).$$

**Proof:** As we mentioned before in Chapter 3, Theorem 4.1 of [3] ensures that

$$\phi_A(t,x,u) = \phi_A(t,e,u)\phi_B(t,x,u).$$

Therefore, the left invariance of the metric allows us to write

$$\begin{aligned} \varrho(\phi_A(t,g,u), \phi_A(t,h,u)) &= \varrho(\phi_A(t,e,u)\phi_B(t,g,u), \phi_A(t,e,u)\phi_B(t,h,u)) \\ &= \varrho(\phi_B(t,g,u), \phi_B(t,h,u)). \end{aligned}$$

This concludes the proof. □

**Corollary 4.10.** *For all $g \in G$, it follows that*

$$\varrho(\phi_A(t,g,u), \phi_A(t,e,u)) = \varrho(\phi_B(t,g,u), e),$$

*where $\phi_B$ is the solution of the associated bilinear system.*

**Proof:** It follows immediately from the above theorem and the fact that the group identity is a singularity of the associated bilinear system. □

For our last result of this section, suppose again that the derivation $\mathcal{D}_{\mathcal{X}_i}$ associated to the linear vector fields $\mathcal{X}_i$ has only eigenvalues with negative real part, $i = 0, 1, \ldots, m$. Under this restrictive hypothesis we can apply the above corollary to show that the trajectories starting at the identity works as an attractor for trajectories associated to certain control functions.

**Corollary 4.11.** *Suppose that the derivations associated to the linear vector fields of the system have only eigenvalues with negative real part. Let $u(t) = (u_1(t), \ldots, u_m(t))$ be a control function such that $u_i(t) \geq 0$. Then*

$$\lim_{t\to\infty} \varrho(\phi_A(t,g,u), \phi_A(t,e,u)) = 0,$$

*for all* $g \in G^- = G_0^- \cap \cdots \cap G_m^-$.

**Proof:** In fact, the previous corollary gives us

$$\varrho(\phi_A(t,g,u), \phi_A(t,e,u)) = \varrho(\phi_B(t,g,u), e).$$

On the other hand, from item 1) of Theorem 4.8 we have that

$$\lim_{t\to\infty} \varrho(\phi_B(t,g,u), e) = 0.$$

This concludes the proof. □

**Remark 4.12.** If we consider that the derivation has only eigenvalues with positive real part, an analogous result is valid for points $g \in G^+ = G_0^+ \cap \cdots \cap G_m^+$.

## 4.4 Dynamical Properties of Control Systems

In this section, we conclude our study giving the notions of stable sets, which were introduced in [8] in the context of semigroup actions.

**Definition 4.13.** *Let* $S \subset G$ *be a subset.*

*1) The set* $S$ *is said to be* $\Sigma$ *- stable if for all* $\epsilon > 0$ *and* $x \in S$*, there is* $\delta$ *(depending on* $x$*) such that the following statement is true:*

*If* $y \in B(x;\delta)$ *then* $\phi(t,y,u) \in B(S;\epsilon)$*, for all control function* $u$ *and* $t \in \mathbb{R}^+$*.*

*2) The set* $S$ *is said to be* $\Sigma$ *- uniformly stable if for all* $\epsilon > 0$ *and* $x \in S$*, there is* $\delta$ *(not depending on* $x$*) such that the following statement is true:*

*If* $y \in B(x;\delta)$ *then* $\phi(t,y,u) \in B(S;\epsilon)$*, for all control function* $u$ *and* $t \in \mathbb{R}^+$*.*

*3) The set* $S$ *is said to be* $\Sigma$ *- orbitally stable if for all* $U$ *neighborhood of* $S$ *there is a positively* $\Sigma$ *- invariant neighborhood* $V$ *of* $S$ *such that* $V \subset U$*.*

Evidently, every $\Sigma$ - uniformly stable set is $\Sigma$ - stable. The converse is not true in general. However, the following result is given in [8, Theorem 3.2], page 238:

**Theorem 4.14.** *Let $S$ be a compact $\Sigma$ - stable set. Then $S$ is $\Sigma$ - uniformly stable.*

Now we show that the study of stable sets of control system on Lie groups comes down to the study of stable sets that contains the group identity.

**Theorem 4.15.** *Let $\Sigma$ be a control system on a lie group $G$ and $S$ be a subset of $G$. The following assertions hold:*

*i) If $\Sigma = \Sigma_I$, then $S$ is $\Sigma_I$ - stable if, and only if, for all $x \in S$, the set $S \cdot x^{-1}$ is $\Sigma_I$ - stable.*

*ii) Suppose that $\Sigma = \Sigma_L$ and that $S$ contains a fixed point $x \in \mathcal{X}$. Then $S$ is $\Sigma_L$ - stable if, and only if, the set $S \cdot x^{-1}$ is $\Sigma_L$ - stable.*

*iii) Suppose that $\Sigma = \Sigma_B$ is a commutative bilinear control system and that $S$ contains a fixed point $x$ of the system (this occurs, in particular, when $x$ is a fixed point of every linear vector field $\mathcal{X}_0$, $\mathcal{X}_0$, $\ldots$, $\mathcal{X}_m$). Then $S$ is $\Sigma_B$ - stable if, and only if, the set $S \cdot x^{-1}$ is $\Sigma_B$ - stable.*

*iv) Suppose that $\Sigma = \Sigma_A$ and that $S$ contains a fixed point $x$ of the associated bilinear system $\Sigma_B$. Then $S$ is $\Sigma_A$ - stable if, and only if, the set $S \cdot x^{-1}$ is $\Sigma_A$ - stable.*

**Proof:** Let $\varrho$ stand for a right invariant riemannian distance on $G$. To prove i), suppose that $S \subset G$ is a $\Sigma_I$ - stable set. Let $x \in S$. We show that $S \cdot x^{-1}$ is also $\Sigma_I$ - stable. Let $yx^{-1} \in S \cdot x^{-1}$ and $\epsilon > 0$. By assumption, there is $\delta > 0$ such that

$$\varrho(z, y) < \delta \implies \phi(t, z, u) \in B(S; \epsilon),$$

for all $t \in \mathbb{R}^+$ and all control function $u$. Now, considering that $\varrho$ is right invariant, we have that

$$\varrho(zx^{-1}, yx^{-1}) < \delta \implies \phi(t, z, u)x^{-1} \in B(S; \epsilon)x^{-1}.$$

The right invariance of the system implies that $\phi(t, z, u)x^{-1} = \phi(t, zx^{-1}, u)$ (see [16], p. 316). Thus,

$$\varrho(zx^{-1}, yx^{-1}) < \delta \implies \phi(t, zx^{-1}, u) \in B(S \cdot x^{-1}; \epsilon).$$

This is equivalent to saying that $S \cdot x^{-1}$ is $\Sigma_I$ - stable and concludes the proof for i). The proof of ii) is similar. We just require this time that $x \in S$ is a fixed point for $\mathcal{X}$. That is, $\varphi_t(x) = x$, for all $t \in \mathbb{R}$. Suppose that $S$ is a $\Sigma_L$ - stable set. As in the previous item, let $yx^{-1} \in S \cdot x^{-1}$ and $\epsilon > 0$. By assumption, there is $\delta > 0$ such that

$$\varrho(z, y) < \delta \implies \phi(t, z, u) \in B(S; \epsilon),$$

for all $t \in \mathbb{R}^+$ and all control function $u$. Again,

$$\varrho(zx^{-1}, yx^{-1}) < \delta \implies \phi(t, z, u)x^{-1} \in B(S; \epsilon)x^{-1}.$$

As $x$ is a fixed point for $\mathcal{X}$, then $x^{-1}$ also is since $\varphi_t$ is an automorphism of G, for all $t \in \mathbb{R}$. Then we have that the property $\phi(t, z, u)x^{-1} = \phi(t, z, u)\varphi_t(x^{-1}) = \phi(t, zx^{-1}, u)$ is valid (see [9, Proposition 2.4], page 356). Thus,

$$\varrho(zx^{-1}, yx^{-1}) < \delta \implies \phi(t, zx^{-1}, u) \in B(S \cdot x^{-1}; \epsilon).$$

Now, to prove iii), let $S$ be a $\Sigma_B$ - stable set that contains a fixed point $x$ of the system. Take a point $yx^{-1} \in S \cdot x^{-1}$. Choose $\delta > 0$ that satisfies

$$\varrho(z, y) < \delta \implies \phi(t, z, u) \in B(S; \epsilon).$$

It follows that

$$\varrho(zx^{-1}, yx^{-1}) < \delta \implies \phi(t, z, u)x^{-1} \in B(S; \epsilon)x^{-1}.$$

By assumption, $x^{-1}$ is a fixed point of the system. Also, the commutativity of the system assures that the solutions are group of automorphisms. This way, we have

$$\phi(t, z, u)x^{-1} = \phi(t, z, u)\phi(t, x^{-1}, u) = \phi(t, zx^{-1}, u).$$

Therefore,

$$\varrho(zx^{-1}, yx^{-1}) < \delta \implies \phi(t, zx^{-1}, u) \in B(S \cdot x^{-1}; \epsilon).$$

Finally, the proof of iv) follows the same idea of the previous items. It is necessary to use again the fact that the solutions of an affine system, with initial condition $x$, has the property

$$\phi_A(t, x, u) = \phi_A(t, e, u)\phi_B(t, x, u).$$

Again, consider a point $y \in S \cdot x^{-1}$ and choose $\delta > 0$ such that

$$\varrho(z, y) < \delta \implies \phi_A(t, z, u) \in B(S; \epsilon).$$

Using the property described above, it follows that

$$\phi_A(t, z, u)x^{-1} = \phi_A(t, z, u)\phi_B(t, x^{-1}, u) = \phi_A(t, zx^{-1}, u).$$

So

$$\varrho(zx^{-1}, yx^{-1}) < \delta \implies \phi_A(t, zx^{-1}, u) \in B(S \cdot x^{-1}; \epsilon),$$

which implies the stability of $S \cdot x^{-1}$ and concludes the proof. □

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