UNIVERSIDADE ESTADUAL DE MARINGÁ
CENTRO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA
(Doutorado)

JULIANA RAUPP DOS REIS SETTI

CONTROL SETS FOR BILINEAR AND AFFINE CONTROL SYSTEMS ON $\mathbb{R}^n$

Maringá-PR
2022

This study was financed by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - grant n°88882449185/2019-01.

UNIVERSIDADE ESTADUAL DE MARINGÁ
CENTRO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

JULIANA RAUPP DOS REIS SETTI

Tese apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática, Centro de Ciências Exatas da Universidade Estadual de Maringá, como requisito para obtenção do título de Doutora em Matemática.
Área de concentração: Geometria e Topologia.

Orientador: Prof. Dr. Alexandre José Santana.

Maringá-PR
2022

Dados Internacionais de Catalogação na Publicação (CIP)
(Biblioteca Setorial BSE-DMA-UEM, Maringá, PR, Brasil)

Setti, Juliana Raupp dos Reis
S495c Control sets for bilinear and affine control systems on Rn / Juliana Raupp dos Reis Setti. -- Maringá, 2022.
114 f. : il.

Orientador: Prof. Dr. Alexandre José Santana.
Tese (doutorado) - Universidade Estadual de Maringá, Centro de Ciências Exatas, Programa de Pós-Graduação em Matemática - Área de Concentração: Geometria e Topologia, 2022.

1. Sistema de controle afim. 2. Sistema de controle bilinear homogêneo. 3. Conjuntos controláveis. 4. Affine control system. 5. Homogeneous bilinear control system. 6. Control sets. I. Santana, Alexandre José, orient. II. Universidade Estadual de Maringá. Centro de Ciências Exatas. Programa de Pós-Graduação em Matemática - Área de Concentração: Geometria e Topologia. III. Título.

CDD 22.ed. 512.55
Edilson Damasio CRB9-1.123

JULIANA RAUPP DOS REIS SETTI

CONTROL SETS FOR BILINEAR AND AFFINE CONTROL SYSTEMS ON R^n

Tese apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática, Centro de Ciências Exatas da Universidade Estadual de Maringá, como parte dos requisitos necessários para a obtenção do título de Doutora em Matemática tendo a Comissão Julgadora composta pelos membros:

COMISSÃO JULGADORA:

Prof. Dr. Alexandre José Santana - UEM (Presidente)
Prof. Dr. Fritz Colonius - Universität Augsburg - Alemanha
Prof. Dr. Mauro Moraes Alves Patrão - UnB
Prof. Dr. João Augusto Navarro Cossich - UTFPR
Prof. Dr. Marcos Roberto Teixeira Primo - UEM
Profa. Dra. Maria Elenice Rodrigues Hernandes - UEM

Aprovada em: 25 de fevereiro de 2022.
Local de defesa: Videoconferência – Google Meet (https://meet.google.com/kob-thpe-ddp)

Dedico este trabalho ao meu esposo
Anderson Macedo Setti.

Acknowledgments

I would like to thank my beloved husband, Anderson Macedo Setti, for being on my side during all this journey, knowing that supporting, isn't sometimes, the easiest part, therefore struggled to comprehend me, and always being pleased to hear me whenever I needed to clarify my ideas, been prepared to read my notes and his suggestions helped me a lot.

Thanks for my mother, Marilda Alhevi Raupp dos Reis and to my parents- in- law Nelson Setti and Maria Elizabete Macedo Maciel, who were always willing to share the emotional support and forgiving me every moment I wasn't present. I would like to thank my colleagues and Professors from the mathematic department, with which I had, for long periods, the opportunity to share ideas, laughs and anguish, in special my colleges Claudia Juliana Fanelli Gonçalves, Priscila Friedemann Cardoso and Suellen Greatti Vieira.

Thanks to Professor Luiz Antônio Barrera San Martin, who, in this difficult long period of the pandemic, volunteered to participate in a weekly seminar on Lie theory. In this seminar he gives us several and important suggestions in Lie theory, in particular in the context of this work. Moreover, this seminar has helped us to maintain motivation and hope for better days.

Thanks to Professor Fritz Colonius, who in practice co-guided this Ph.D dissertation. He dedicated generously and extensively for the development of this work and his contribution was essential.

Thanks to my advisor, Professor Alexandre José Santana, who promptly accepted my request for guidance and was available and able when he was needed, with whom, I could learn more about Mathematics, and also about life, and became for me, an example of person and professional.

I am thankful to the Professors Fritz Colonius, João Augusto Navarro Cossich, Josiney Alves de Souza, Lucas Conque Seco Ferreira, Marcos Roberto Teixeira Primo, Maria Elenice Rodrigues Hernandes and Mauro Moraes Alves Patrão for accepting to participate in the defense committee of this work.

Thanks to the graduate program in mathematics at the State University of Maringá, PMA-UEM, for all the conditions provided for my doctorate.

Finally, I would like to thank the Brazilian research funding institution, CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) for the financial support which allowed my exclusive dedication to the doctoral period.

ABSTRACT

In this thesis we present our study of control sets with nonvoid interior for homogeneous bilinear control systems and affine control systems defined on $\mathbb{R}^n$, considering that the control range is a bounded set.

For homogeneous bilinear control systems the control sets are characterized using the Lie algebra rank condition for the induced system on projective space. This is based on a classic Diophantine approximation result. For affine control systems we start by studying the control sets around the equilibrium points, with more attention to the case where these sets are unbounded. Then, we started to consider periodic trajectories, so our study started to consider spectral properties. For hyperbolic systems, we prove that there is a unique control set with a nonvoid interior, and if the system is uniformly hyperbolic then it is bounded. For nonhyperbolic systems we prove that every control set with a nonvoid interior is unbounded.

We induce a system in projective space and study some relation between control sets and chain control sets of an affine control system and its homogeneous part.

Keywords: affine control system, homogeneous bilinear control system, control set.

RESUMO

Nesta tese apresentamos nosso estudo sobre conjuntos controláveis com interior não vazio para sistemas de controle bilinear homogêneo e sistemas de controle afim definidos em $\mathbb{R}^n$, considerando que a imagem de controle é um conjunto limitado.

Para sistemas bilineares homogêneos os conjuntos controláveis são caracterizados usando a condição do posto da álgebra de Lie para o sistema induzido no espaço projetivo. Isto é baseado em um resultado clássico de aproximação Diofantina. Para sistemas de controle afim nós começamos estudando os conjuntos controláveis em torno dos pontos de equilíbrio, com maior atenção para o caso em que estes conjuntos são ilimitados. Em seguida, passamos a considerar trajetórias periódicas, assim nosso estudo passou a considerar propriedades espectrais. Para sistemas hiperbólicos, provamos que existe um único conjunto controlável com interior não vazio, e se o sistema for ainda uniformemente hiperbólico então este é limitado. E para sistemas não hiperbólicos nós provamos que todo conjunto controlável com interior não vazio é ilimitado.

Nós induzimos um sistema no espaço projetivo e estudamos algumas relações entre conjuntos controláveis e conjuntos controláveis por cadeias de um sistema de controle afim e sua parte homogênea.

Palavras-chave: sistema de controle afim, sistema de controle bilinear homogêneo, conjuntos controláveis.

Contents

List of Figures

Introdution

We study controllability properties for affine control systems of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t)(B_i x(t) + c_i) + d, \quad u(t) \in \Omega, \tag{1}$$

where $A, B_1, \ldots, B_m \in \mathbb{R}^{n\times n}$ and $c_1, \ldots, c_m, d$ are vectors in $\mathbb{R}^n$. The controls $u = (u_1, \ldots, u_m)$ have values in a bounded set $\Omega \subset \mathbb{R}^m$. The set of admissible controls is $\mathcal{U} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \mid u(t) \in \Omega \text{ for almost all } t\}$ or the set $\mathcal{U}_{pc}$ of all piecewise constant functions defined on $\mathbb{R}$ with values in Ω. We also write (1) as

$$\dot{x}(t) = A(u(t))x(t) + Cu(t) + d, \quad u(t) \in \Omega,$$

with $A(u) := A + \sum_{i=1}^{m} u_i B_i, u \in \Omega$, and $C := (c_1, \ldots, c_m)$.

Controllability properties of bilinear and affine systems have been studied since more than fifty years. The first contributions were given by Rink and Mohler [35] who took the set of equilibria as a starting point for establishing results on complete controllability, it contains sufficient conditions for complete controllability and many applications of bilinear control systems. On the other hand, Lie-algebraic methods have yielded important insights. The monograph Elliott [21] emphasizes the use of matrix Lie groups and Lie semigroups and contains a wealth of results on the control of bilinear control systems. A classical result due to Jurdjevic and Sallet [29, Theorem 2] shows that affine system (1) is controllable on $\mathbb{R}^n$ if it has no fixed points and its homogeneous part, the bilinear system

$$\dot{x}(t) = A(u(t))x(t), \quad u(t) \in \Omega,$$

is controllable on $\mathbb{R}^n \setminus \{0\}$. Only the first condition is necessary, certainly it is not sufficient. Initially, bilinear control systems were considered as "nearly linear" (cf. Bruni, Di Pillo, and Koch [8] containing also many early references). However it turned out that characterizing controllability of such systems (even with unrestricted controls) is a very difficult problem. As Jurdjevic [27, p.182] emphasizes, the controllability properties of affine systems are substantially richer and may require "entirely different geometrical considerations" (based on Lie-algebraic methods).

Concerning the literature on controllability properties of affine and bilinear systems many results are found in Mohler [32] and Jurdjevic [27], other contributions are based on the theory of semigroups in Lie groups, this includes Boothby and Wilson [6], Bonnard [4], Jurdjevic and Kupka [28], Gauthier and Bornard [24], Bonnard, Jurdjevic, Kupka, and Sallet [5], Jurdjevic and Sallet [29], San Martin [38].

The main result of Do Rocio, Santana, and Verdi [20, Theorem 1.3] concerns a connected semigroup S with nonvoid interior in an affine group $G = B \rtimes V$, where V is a finite dimensional vector space and B is a semisimple Lie group that acts transitively on $V \setminus \{0\}$. If the linear action of the canonical projection $\pi(S)$ on B is transitive on $V \setminus \{0\}$, then the affine action of S on V is transitive. This improves an earlier result in [29]. An application to an affine control system of the form

$$\dot{x} = Ax + a + uBx + ub \text{ with } u \in \mathbb{R}, \tag{2}$$

where $A, B \in \mathfrak{sl}(2, \mathbb{R})$ and $a, b \in \mathbb{R}^2$, results in a sufficient controllability criterion in terms of these parameters.

Answering a question by Sachkov [37], Do Rocio, San Martin, and Santana [19] prove that systems of the form (2) with $a = b = 0$ and unrestricted control may not be completely controllable on $\mathbb{R}^n \setminus \{0\}$ while there is no nontrivial proper closed convex cone in $\mathbb{R}^n$ which is positively invariant. For the relation to the results in the present paper see Remark 2.2.17 and also Proposition 2.2.18.

Motivated by Kalman criterion for controllability of linear systems, an early goal was to show that the controllability of bilinear control systems (without control restrictions) has an algebraic characterization. This hope did not bear out, in spite of many partial results. The present work is mainly concerned with the analysis of control sets that is which are maximal subsets of complete approximate controllability in $\mathbb{R}^n$; cf. Definition 1.1.7 and Colonius and Kliemann [13] for a general theory.

Our results on control sets will also yield some results on controllability on $\mathbb{R}^n$. In our results about control set with nonvoid interior of homogeneous bilinear controls system and control set with nonvoid interior around equilibria of affine control system, we do not restrict our attention to the situation where the system semigroup has nonvoid interior in the system group. Correspondingly, our results are not based on methods for semigroups in Lie groups. But to progress in the study of control set with nonvoid interior we started to consider periodic solution, here the spectral properties and semigroups with nonvoid interior become crucial.

In the first part of this work we discuss control sets for homogeneous bilinear systems which are a special case of (1) with $c_1 = \cdots = c_m = d = 0$. It is well known that, for this class of systems, one can separate controllability properties into properties concerning the angular part on the unit sphere $\mathbb{S}^{n-1}$ and the radial part. In particular, by Bacciotti and

Vivalda [3, Theorem 1] the induced system on projective space $\mathbb{P}^{n-1}$ is controllable if and only if the induced system on $\mathbb{S}^{n-1}$ is controllable.

Theorem 2.2.2 shows that every control set ${}_{\mathbb{S}}D$ with nonvoid interior on $\mathbb{S}^{n-1}$ induces a control set D on $\mathbb{R}^n \setminus \{0\}$ given by the cone generated by ${}_{\mathbb{S}}D$, providing that exponential growth and decay can be achieved. Here we use a classical result on Diophantine approximations which allows us to require only the accessibility rank condition on $\mathbb{S}^{n-1}$ in the interior of ${}_{\mathbb{S}}D$. This result is illustrated by two-dimensional examples. For systems satisfying the accessibility rank condition on projective space, the control sets on the unit sphere and on $\mathbb{R}^n \setminus \{0\}$ are characterized in Theorem 2.2.11 and Theorem 2.2.15, respectively. We remark that under the accessibility rank condition in $\mathbb{R}^2$, a complete description of the control sets and of controllability is given in Ayala, Cruz, Kliemann, and Laura-Guarachi [1]. Corollary 2.2.20 characterizes controllability on $\mathbb{R}^n \setminus \{0\}$ for systems satisfying only the accessibility rank condition on $\mathbb{P}^{n-1}$ using a recent result by Cannarsa and Sigalotti [9, Theorem 1] which shows that here approximate controllability implies controllability.

In the second part we analyze control sets for general affine systems. We started with the relation between control set and equilibria. If the systems linearized about equilibria are controllable, Theorem 3.2.6 shows that any pathwise connected set of equilibria is contained in a control set. Additional assumptions on spectral properties of the matrices $A(u) = A + \sum_{i=1}^{m} u_i B_i,\ u \in \Omega$, allow us to get more detailed information. In particular, if 0 is an eigenvalue of $A(u^0)$ for some $u^0 \in \Omega$, one finds an unbounded control set; cf. Theorem 3.2.15. The main open problem for control sets of affine systems if every control set contains an equilibrium.

Posteriorly we consider periodic solutions and divided this study into two cases hyperbolic and nonhyperbolic. In the hyperbolic case (cf. Definition 3.3.1) Theorem 3.3.4 shows that an affine control system has a unique control set D with nonvoid interior. In the uniformly hyperbolic case (cf. Definition 3.3.6) Theorems 3.3.8 and 3.3.11 show that D is bounded and the closure of D is the unique bounded chain control set (here small jumps in the trajectories are allowed; cf. Definition 1.1.12). Hence these systems enjoy similar controllability properties as linear control systems of the form $\dot{x} = Ax + Bu$ with hyperbolic matrix A (i.e., A has no eigenvalues on the imaginary axis), controllable pair (A, B), and compact and convex control range Ω containing 0; cf. Colonius and Kliemann [13, Example 3.2.16]. We remark that, in another direction, control sets of linear control systems on Lie groups have been studied by Da Silva [17] and Ayala and Da Silva [2].

Nonhyperbolic affine systems may possess several control sets with nonvoid interior. By Theorem 3.5.1 each of them is unbounded. The proof takes up ideas from Rink and Mohler [35], replacing the set of equilibria by the set of periodic solutions. Then we compactify the state space using an embedding into projective space $\mathbb{P}^n$ to show that for a control set with nonvoid interior the "boundary at infinity" (cf. Definition 3.4.4) intersects a chain control set of the projectivized homogeneous part; cf. Theorems 3.5.5 and Theorem 3.5.7. Already the special case of linear control systems discussed in the beginning of Section 3.5 shows that here chain transitivity (a classical notion in the theory of dynamical systems; cf. Robinson [36]) plays an important role. An example of an affine system is considered where more than one chain control set of the homogeneous part is contained in the boundary at infinity of a control set. We emphasize that the boundary at infinity concerns the *lines* which are obtained in the limit for large states. Hence one may expect that one obtains relations to the chain control sets of the homogeneous part in projective space $\mathbb{P}^{n-1}$, not primarily in $\mathbb{R}^n \setminus \{0\}$. The main result on the nonhyperbolic case is Theorem 3.5.12 showing that there is a single chain control set in $\mathbb{P}^n$ containing the images of all control sets D with nonvoid interior in $\mathbb{R}^n$. The boundary at infinity of this chain control set contains all chain control sets of the homogeneous part having nonvoid intersection with the boundary at infinity of one of the control sets D. These results cast new light on the relations between affine systems and their homogeneous parts and are intuitively appealing, since one may expect that for unbounded x-values the inhomogeneous part $Cu(t) + d$ becomes less relevant (recall that we assume that Ω is bounded).

The contents of this work are as follows. Section 1.1 describes basic properties of nonlinear control systems and control sets, in the Section 1.2 we present some definition and properties about affine control system on $\mathbb{R}^n$. Furthermore, we state some classical properties of periodic solutions for inhomogeneous linear differential equations. Section 1.3 analyzes the interior of system semigroups, and Section 1.4 applies this to affine control systems and their homogeneous parts. Section 2.1 discusses homogeneous bilinear control systems using their projection to the unit sphere. In Section 2.2 state some results on spectral properties and controllability for homogeneous bilinear control systems. Section 3.1 briefly describes equilibria of affine systems and Section 3.2 presents results on control sets around such equilibria. Section 3.3 shows that for hyperbolic systems a unique control set exists and if the system is uniformly hyperbolic it is bounded. Section 3.4 prepares the discussion of nonhyperbolic systems by describing the embedding of affine control systems into homogeneous bilinear systems and associated systems in projective spaces. Finally, Section 3.5 describes the control sets and their boundaries at infinity for nonhyperbolic affine systems.

Chapter 1

Control system

In this chapter we introduce definitions and properties about control systems. In Section 1.1 we start with the definition of control systems defined in smooth manifolds, we also present here the definitions of control set and chain control set. Then, in Section 1.2, we deal with affine control systems defined in $\mathbb{R}^n$, and some special cases of these systems, in this section we also deal with periodic solutions for a differential equation, this will be used again and again throughout the work. In Section 1.3, we focus on the study of the system group and semigroup. In Section 1.4 we continue to deal with the group and semigroup of an control system, but for the specific case of an affine control system in $\mathbb{R}^n$.

1.1 Basic properties of control systems

In this section we present some definitions and properties about control system defined on a smooth manifold. Here are defined control system, control set of control system, chain control set and other objects. For the general theory of control systems we refer to Jurdjevic [27] and Sontag [39].

In this work we consider the class of the control system defined as follows.

Definition 1.1.1. *Let M be a smooth manifold, Ω a convex compact nonvoid subset of $\mathbb{R}^m$, with $0 \in \Omega$, $f_0, \ldots, f_m$ smooth vector fields on M,*

$$\mathcal{U} = \{u : \mathbb{R} \longrightarrow \Omega \mid u \text{ is locally integrable}\}$$

and $F : M \times \Omega \longrightarrow TM$ the differentiable map defined by

$$F(u, x) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x),$$

where $u(t) = (u_1(t), \ldots, u_m(t)) \in \Omega$. *Then the family of differential equations*

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{1.1}$$

is called a ***control system*** *and* $\mathcal{U}$ *is called the* ***family of admissible control****, and its elements* ***control functions****. The set* Ω *is called* ***control range*** *and its elements* ***control values****. The manifold* M *is called the* ***state space*** *and the function* F *the* ***right-hand side*** *of the control system.*

Here we will consider that the vector fields $f_0, \ldots, f_m$ are complete.

For the general theory see also Kawan [30], where a theorem about the approximation of arbitrary trajectories of a control system by trajectories corresponding to piecewise constant control functions is proved, for this type of system. For this reason, we will consider $\mathcal{U}$ the set of all piecewise constant controls defined on $\mathbb{R}$ with value in Ω, below we clarify what we consider as a piecewise constant controls.

A control function u is called **piecewise constant control** if $\mathbb{R}$ is decomposed into subintervals limited inferiorly such that u is constant in each of these intervals. The set of all piecewise constant control is denoted by $\mathcal{U}_{pc}$.

We assume that for each initial state $x \in M$ and every control function $u \in \mathcal{U}_{pc}$, there exists a unique solution of (1.1) denoted by $\varphi(t, x, u)$, defined for all $t \in \mathbb{R}$, and satisfies $\varphi(0, x, u) = x$ depending continuously on x.

Definition 1.1.2. *The shift flow* Θ *is defined by*

$$\begin{array}{rccc} \Theta: & \mathbb{R} \times \mathcal{U}_{pc} & \longrightarrow & \mathcal{U}_{pc} \\ & (t, u) & \longmapsto & \Theta_t u \end{array}$$

where $\Theta_t u(s) = u(t+s)$ *for all* $s \in \mathbb{R}$.

As the solution of (1.1) are defined on $\mathbb{R}$ it is possible to define a map

$$\begin{array}{rccc} \varphi: & \mathbb{R} \times M \times \mathcal{U} & \longrightarrow & M \\ & (t, x, u) & \longmapsto & \varphi(t, x, u) \end{array},$$

called **transition map** that satisfy

$$\varphi(t+s, x, u) = \varphi(t, \varphi(s, x, u), \Theta_s u) \tag{1.2}$$

what is called the **cocycle property**, this is proven in Kawan [30, Proposition 1.2.8]. Moreover, it holds that for all $t_1, t_2 \geq 0$ and $u_1, u_2 \in \mathcal{U}_{pc}$

$$\varphi(t_2, \varphi(t_1, x, u_1), u_2) = \varphi(t_1 + t_2, x, u),$$

where

$$u(t)=\begin{cases} u_1(t), \text{ for } t\in[0,t_1] \\ u_2(t-t_1), \text{ for } t\in(t_1,t_2+t_1] \end{cases}.$$

For each function u, the map $\varphi_u:\mathbb{R}\times M\longrightarrow M$ is continuous and for $t\in\mathbb{R}$ the map $\varphi_{t,u}:M\longrightarrow M$ is a homeomorphism, see Kawan [30, Proposition 1.1.10 and Corollary 1.2.12].

If we assume that Ω is convex and $\mathcal{U}$ is a subset of all the essentially bounded functions, then $\mathcal{U}$ is compact and metrizable in the weak*-topology of $\mathrm{L}^\infty(\mathbb{R},\mathbb{R}^m)$ and both the maps Θ and φ are continuous with respect to this topology.

Proposition 1.1.3. *Consider the control system (1.1). Let $\mathcal{U}_{pc}$ be endowed with the weak*-topology. Then the following assertions are holds:*

(i) $\mathcal{U}_{pc}$ is a compact, separable metrizable space.

(ii) The maps Θ and φ are continuous.

(iii) The mapping

$$\begin{array}{rccc} \Psi: & \mathbb{R}\times M\times\mathcal{U}_{pc} & \longrightarrow & M\times\mathcal{U}_{pc} \\ & (t,x,u) & \longmapsto & (\varphi(t,x,u),\Theta_t u) \end{array}$$

is a continuous flow, called ***control flow of system (1.1)*** .

Proof. A demonstration of this result can be seen in Kawan [30, Proposition 1.3.14] . □

Now we will define reachable sets they are objects of great importance in our study.

Definition 1.1.4. *The set of points reachable from $x\in M$ and controllable to $x\in M$ up to time $T>0$ are defined, respectively, by*

$$\begin{aligned} \mathcal{O}^+_{\leq T}(x) &:= \{y\in M \mid \textit{there are } 0\leq t\leq T \textit{ and } u\in\mathcal{U}_{pc} \textit{ with } y=\varphi(t,x,u)\}, \\ \mathcal{O}^-_{\leq T}(x) &:= \{y\in M \mid \textit{there are } 0\leq t\leq T \textit{ and } u\in\mathcal{U}_{pc} \textit{ with } x=\varphi(t,y,u)\}. \end{aligned}$$

*Therefore, the reachable set (or "**positive orbit**") from x, and the set controllable to x (or "**negative orbit**" of x) respectively, are*

$$\begin{aligned} \mathcal{O}^+(x) &= \bigcup\nolimits_{T>0}\mathcal{O}^+_{\leq T}(x), \\ \mathcal{O}^-(x) &= \bigcup\nolimits_{T>0}\mathcal{O}^-_{\leq T}(x). \end{aligned}$$

The system (1.1) is called locally accessible in x, if $\mathcal{O}^+_{\leq T}(x)$ and $\mathcal{O}^-_{\leq T}(x)$ have nonvoid interior for all $T>0$ and the system is called ***locally accessible*** *if this holds in every point $x\in M$.*

In some results, we will need of the following rank condition, because in that case the system is locally accessible.

Lemma 1.1.5. *The system (1.1) is locally accessible if the following condition is holds*

$$\dim \mathcal{LA}\{f_0, f_1, \ldots, f_m\}(x) = \dim M, \ \textit{for all } x \in M, \tag{1.3}$$

where $\mathcal{LA}\{f_0, f_1, \ldots, f_m\}(x)$ is the subspace of the tangent space T_xM corresponding to the vector fields, evaluated in x, in the Lie algebra generated by $f_0, f_1, \ldots, f_m$. For analytic systems on an analytic manifold this condition is also necessary for local accessibility.

The condition (1.3) is know as ***accessibility rank condition*** *.*

Proof. A prove for this lemma can be view in Colonius and Kliemann [13, Theorem A.4.4 and Theorem A.4.6]. □

Here we present some properties about positive orbit and negative orbit that made some later proofs simpler.

Property 1.1.6. *The reachable set and controllable set satisfy the following properties:*

(i) If $x \in \mathcal{O}^+(y)$ so $\mathcal{O}^+(x) \subset \mathcal{O}^+(y)$.

(ii) If $x \in \mathcal{O}^+(y)$ and $y \in \mathcal{O}^+(z)$, then $x \in \mathcal{O}^+(z)$.

(iii) If $x \in \mathcal{O}^-(y)$ so $\mathcal{O}^-(x) \subset \mathcal{O}^-(y)$.

(iv) If $x \in \mathcal{O}^-(y)$ and $y \in \mathcal{O}^-(z)$, then $x \in \mathcal{O}^-(z)$.

(v) If $y \in \overline{\mathcal{O}^+(x)}$ and $y \in \operatorname{int}(\mathcal{O}^-(y))$ then $y \in \mathcal{O}^+(x)$.

Proof. (i) As $x \in \mathcal{O}^+(y)$ there are $T_1 \geq 0$ and $u \in \mathcal{U}_{pc}$ such that $\varphi(T_1, y, u) = x$. For every $z \in \mathcal{O}^+(x)$ there are $T_2 \geq 0$ and $v \in \mathcal{U}_{pc}$ such that $\varphi(T_2, x, v) = z$. Consider the control w defined by $w(t) = u(t)$ if $t \in (-\infty, T_1]$ and $w(t) = v(t - T_1)$ if $t \in (T_1, \infty)$ so

$$\begin{aligned}
\varphi(T_1 + T_2, y, u) &= \varphi(T_2, \varphi(T_1, y, w), \Theta_{T_1} w) \\
&= \varphi(T_2, \varphi(T_1, y, u), \Theta_{T_1} w) \\
&= \varphi(T_2, x, w(T_1 + \cdot)) \\
&= \varphi(T_2, x, v) = z,
\end{aligned}$$

so $z \in \mathcal{O}^+(y)$. The item (ii) follow of (i) and items (iii) and (iv) are analogous to these.

For (v) consider $(\varphi(t_n, x, u_n))_{n \in \mathbb{N}}$ a sequence such that $\varphi(t_n, x, u_n) \to y$ when $n \to \infty$. As $\operatorname{int}(\mathcal{O}^-(y))$ is a neighborhood of y there are $\varphi(t_{n_0}, x, u_{n_0}) \in \operatorname{int}(\mathcal{O}^-(y))$, so $y \in \mathcal{O}^+(\varphi(t_{n_0}, x, u_{n_0})) \subset \mathcal{O}^+(x)$. □

A system is said to be **controllable** if given any two states $x, y \in M$ we can go from x to y through solutions, that is, there are positive times $t_1, t_2 \in \mathbb{R}$ and control functions $u, v \in \mathcal{U}$ such that $\varphi(t_1, x, u) = y$ and $\varphi(t_2, y, v) = x$. This property is extensively investigated in the literature where we seek to determine conditions for it to be true. However this is not the objective of this work. Here we are interested in investigating subsets where there are properties very close to controllability, the subsets of complete approximate controllability, which are introduced in the next definition.

Definition 1.1.7. *A nonvoid set $D \subset M$ is called a **control set** of system (1.1) if it has the following properties:*

(i) for all $x \in D$ there is a control function $u \in \mathcal{U}_{pc}$ such that $\varphi(t, x, u) \in D$, for all $t \geq 0$,

(ii) for all $x \in D$ one has $D \subset \overline{\mathcal{O}^+(x)}$, and

(iii) D is maximal with these, that is, if $D' \supset D$ satisfies conditions (i) and (ii), then $D' = D$.

*A control set $D \subset M$ is called an **invariant control set** if $\overline{D} = \overline{\mathcal{O}^+(x)}$, for all $x \in D$. All other control sets are called variant.*

If the intersection of two control sets is nonvoid, the maximality property (iii) implies that they coincide.

The next result about control sets with nonvoid interior, can be found in Colonius and Kliemann [13]. We present it here to facilitate your consultation.

Lemma 1.1.8. *Let D be a control set of system (1.1) with nonvoid interior.*

(i) If $y \in \operatorname{int}(D)$ is locally accessible, then $y \in \mathcal{O}^+(x)$ for all $x \in D$.

(ii) If the system is locally accessible from all $y \in \operatorname{int}(D)$, then $\operatorname{int}(D) \subset \mathcal{O}^+(x)$ for all $x \in D$, and for every $y \in \operatorname{int}(D)$ one has $D = \overline{\mathcal{O}^+(y)} \cap \mathcal{O}^-(y)$.

Proof. See Colonius and Kliemann [13, Lemma 3.2.13] . □

Lemma 1.1.9. *Suppose that local accessibility holds. If $x \in \operatorname{int}(\mathcal{O}^-(x)) \cap \operatorname{int}(\mathcal{O}^+(x))$ then $D = \mathcal{O}^-(x) \cap \overline{\mathcal{O}^+(x)}$ is a control set and $x \in \operatorname{int}(D)$.*

Proof. By the assumption, there is a neighborhood N of x with $N \subset \operatorname{int}(\mathcal{O}^-(x)) \cap \operatorname{int}(\mathcal{O}^+(x))$. For every $z \in N$ there are $T_1, T_2 > 0$, $u_1, u_2 \in \mathcal{U}_{pc}$ such that $\varphi(T_1, z, u_1) = x$ and $\varphi(T_2, x, u_2) = z$. Define the control function

$$v(t) = \begin{cases} u_1(t), & if\ t \in [0, T_1) \\ u_2(t - T_1), & if\ t \in [T_1, \infty) \end{cases}.$$

Note that $\varphi(\cdot, z, v)$ is $T_1 + T_2$-periodic. In fact,

$$\begin{aligned} \varphi(t + T_1 + T_2, z, v) &= \varphi(t + T_2, \varphi(T_1, z, v), \Theta_{T_1} v) \\ &= \varphi(t + T_2, x, \Theta_{T_1} v) \\ &= \varphi(t, \varphi(T_2, x, \Theta_{T_1} v), \Theta_{T_1+T_2} v) \\ &= \varphi(t, z, v). \end{aligned}$$

Thus $\varphi(t, z, v) \in \mathcal{O}^-(x) \cap \mathcal{O}^+(x)$ for all $t \in \mathbb{R}$, so N satisfies the condition (i) of the Definition 1.1.7. Moreover $x \in \overline{\mathcal{O}^+(z)}$, so $\overline{\mathcal{O}^+(x)} \subset \overline{\mathcal{O}^+(z)}$, thus $N \subset \overline{\mathcal{O}^+(z)}$, therefore N satisfies the condition (ii) of the Definition 1.1.7.

Therefore, N is contained in some control set D, by Lemma 1.1.8 (ii) $D = \mathcal{O}^-(x) \cap \overline{\mathcal{O}^+(x)}$. □

Next we introduce a notion of controllability allowing for (small) jumps between pieces of trajectories. Here we fix a metric d on M.

Definition 1.1.10. *Fix $x, y \in M$ and let $\varepsilon, T > 0$. A* ***controlled*** (ε, T)***-chain*** *from x to y is given by $n \in \mathbb{N}$, $x_0, \ldots, x_n \in M$, $u_0, \ldots, u_{n-1} \in \mathcal{U}$, $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$, and*

$$d(\varphi(t_j, x_j, u_j), x_{j+1}) \leq \varepsilon, \ \textit{for all } j = 0, \ldots, n-1.$$

If for every $\varepsilon, T > 0$ there is a controlled (ε, T)-chain from x to y, then the point x is chain controllable to y.

Proposition 1.1.11. *The concatenation of two controlled (ε, T)-chains again yields a controlled (ε, T)-chain.*

Proof. Suppose there are a controlled (ε, T)-chain from x to y and a controlled (ε, T)-chain from y to z, that is, there are $n \in \mathbb{N}$, $x_0, \ldots, x_n \in M$, $u_0, \ldots, u_{n-1} \in \mathcal{U}$, $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$, and

$$d(\varphi(t_j, x_j, u_j), x_{j+1}) \leq \varepsilon, \ \text{for all } j = 0, \ldots, n-1,$$

and $m \in \mathbb{N}$, $y_0, \ldots, y_m \in M$, $v_0, \ldots, v_{m-1} \in \mathcal{U}$, $s_0, \ldots, s_{m-1} \geq T$ with $y_0 = y$, $y_m = z$, and

$$d(\varphi(s_j, y_j, v_j), y_{j+1}) \leq \varepsilon, \ \text{for all } j = 0, \ldots, m-1.$$

Consider $n + m \in \mathbb{N}$, $x_0, \ldots, x_n, x_{n+1} = y_1, x_{n+2} = y_2, \ldots, x_{m+n} = z$, controls function $u_0, \ldots, u_{n-1}, u_n = v_0, u_{n+1} = v_1, \ldots, u_{n+m-1} = v_{m-1} \in \mathcal{U}$, and times $t_0, \ldots, t_{n-1}, t_n = s_0, t_{n+1} = s_1, \ldots, t_{n+m-1} = s_{m-1} \geq T$ is a controlled (ε, T)-chain from x to z.

$\square$

In analogy to control sets, chain control sets are defined as maximal regions of chain controllability.

Definition 1.1.12. *A set $E \subset M$ is called a* ***chain control set*** *of system (1.1) if it has the following properties:*

(i) for all $x \in E$ there is $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$,

(ii) for all $x, y \in E$ and $\varepsilon,\ T > 0$ there is a controlled (ε, T)-chain from x to y, and

(iii) E is maximal (with respect to set inclusion) with these properties.

Chain control sets are closed and for locally accessible systems every control set with nonvoid interior is contained in a chain control set. For the properties of control sets and chain control sets stated above, we refer to Colonius and Kliemann [13, Chapters 3 and 4]. Where it is also shown that, for almost all control ranges, a chain control set coincides with the closure of a control set with nonvoid interior, if the so-called "inner-pair condition" holds. Hyperbolicity conditions for the associated control flow yield a similar result based on a shadowing lemma; cf. Colonius and Du [12]. Da Silva and Kawan [18] also exploit hyperbolicity conditions for chain control sets.

Definition 1.1.13. *A compact subset K of a continuous dynamical system $\psi : \mathbb{R} \times M \to M$ on a metric space M is called* ***chain transitive****, if for all $x, y \in K$ and all $\varepsilon, T > 0$ there is an (ε, T)-chain from x to y given by $n \in \mathbb{N}$, $x_0 = x, x_1, \ldots, x_n = y \in M$, and $t_0, \ldots, t_{n-1} \geq T$ with*

$$d(\psi(t_j, x_j), x_{j+1}) \leq \varepsilon, \ \textit{for all } j = 0, \ldots, n-1.$$

It follows from of the Proposition 1.1.3 that the control system (1.1) define a dinamical system. By [13, Theorem 4.3.11] a compact chain control set E gives rise to a maximal chain transitive set $\mathcal{E}$ of this flow via

$$\mathcal{E} = \{(u, x) \in \mathcal{U} \times E \mid \varphi(t, x, u) \in E \text{ for all } t \in \mathbb{R}\}. \tag{1.4}$$

1.2 Affine control system on $\mathbb{R}^n$

Our aim is to study control sets for affine control systems on $\mathbb{R}^n$, for this study we introduce the following notations. Here we also recall some properties about periodic solutions of periodic differential equation.

We will study properties of affine control systems of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t)(B_i x(t) + c_i) + d, \tag{1.5}$$

where $A, B_1, \ldots, B_m \in \mathbb{R}^{n\times n}$ and $c_1, \ldots, c_m, d$ are vectors in $\mathbb{R}^n$. The controls $u = (u_1, \ldots, u_m)$ have values in a set $\Omega \subset \mathbb{R}^m$.

Frequently, we abbreviate

$$A(u) := A + \sum_{i=1}^{m} u_i B_i \text{ for } u \in \Omega \text{ and } C := (c_1, \ldots, c_m) \in \mathbb{R}^{n\times m},$$

hence the columns of C are given by the c_i. Then (1.5) can be written as

$$\dot{x}(t) = A(u(t))x(t) + Cu(t) + d.$$

Bilinear control systems are special cases and obtained when $d = 0$, i.e.,

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t)(B_i x(t) + c_i) = A(u(t))x(t) + Cu(t).$$

Other special cases are homogeneous bilinear control systems, given of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t)B_i x(t) = A(u(t))x(t). \tag{1.6}$$

Given an affine control system (1.5), we can define a homogeneous bilinear control system get $C = 0$ and $d = 0$, this last system is called **homogeneous part** of the system (1.5).

For fixed control $u \in \mathcal{U}_{pc}$ (1.5) is a nonautonomous inhomogeneous linear differential equation. Denote by $\Phi_u(t, s) \in \mathbb{R}^{n\times n}$ the **principal matrix solution**, i.e., the solution of

$$\frac{d}{dt}\Phi_u(t, s) = A(u(t))\Phi_u(t, s), \quad \Phi_u(s, s) = I.$$

The solutions $\varphi(t, x_0, u), t \in \mathbb{R}$, of (1.5) with initial condition $\varphi(0, x_0, u) = x_0 \in \mathbb{R}^n$ are given by

$$\varphi(t, x_0, u) = \Phi_u(t, 0)x_0 + \int_0^t \Phi_u(t, s)[Cu(s) + d]ds, \quad t \in \mathbb{R},$$

and, in particular, the solutions of (1.6) are

$$\varphi(t, x_0, u) = \Phi_u(t, 0)x_0, \quad t \in \mathbb{R}.$$

This readily implies for $\alpha \in \mathbb{R}$

$$\varphi(t, \alpha x_0, u) = \Phi_u(t, 0)\alpha x_0 = \alpha\varphi(t, x_0, u). \tag{1.7}$$

In other literature the matrix $\Phi_u(t, s)$ is denoted by $exp((t-s)A(u))$, but in literature used as reference here we find this notation, so let is use it.

For a matrix A we will denote the **spectrum of** A by

$$\text{spec}(A) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is eigenvalue of } A\},$$

and the **real generalized eigenspace** for an eigenvalue λ of A, by $\mathbf{E}(A, \lambda)$, that is, the subspace

$$\mathbf{E}(A, \lambda) = \left\{v \in \mathbb{R}^n \mid (A - \lambda I)^k v = 0, \text{ for some } \ 0 < k \in \mathbb{N}\right\}.$$

Throughout this text we will often need to consider periodic equations and so we recall some basic facts on periodic solutions of inhomogeneous periodic differential equations following Hahn [25, § 72]. Consider

$$\dot{x}(t) = P(t)x(t) + z(t), \tag{1.8}$$

where $P(\cdot) \in L^\infty(\mathbb{R}, \mathbb{R}^{n\times n})$ and $0 \neq z(\cdot) \in L^\infty(\mathbb{R}, \mathbb{R}^n)$ are τ-periodic, i.e., $P(t+\tau) = P(t)$ and $z(t+\tau) = z(t)$ for all $t \in \mathbb{R}$.

The principal matrix solution $\Phi(t, s) \in \mathbb{R}^{n\times n}, t, s \in \mathbb{R}$, is given by

$$\frac{d}{dt}\Phi(t, s) = P(t)\Phi(t, s) \text{ with } \Phi(s, s) = I.$$

It satisfies $\Phi(t, r) = \Phi(t, s)\Phi(s, r)$ for $t, s, r \in \mathbb{R}$.

The homogeneous equation

$$\dot{x}(t) = P(t)x(t) \tag{1.9}$$

has a τ-periodic solution (not unique) if and only if 1 is an eigenvalue of $\Phi(\tau, 0)$; cf. Hahn [25, Theorem 60.3]. Denote the solutions of the homogeneous equation (1.9) by $\psi\left(t, x^0\right), t \in \mathbb{R}$, with $\psi\left(0, x^0\right) = x^0$. The numbers $\lambda_1, \ldots, \lambda_n$,

$$\lambda_j := \lim_{t\to\pm\infty} \frac{1}{t} \log \left\|\psi(t, x^0)\right\| \text{ for } x^0 \neq 0, \tag{1.10}$$

are the **Floquet exponents**. The **Floquet multipliers** $\rho_j \in \mathbb{C}$ are the eigenvalues of $\Phi(\tau, 0)$ and the Floquet exponents satisfy $\lambda_j = \frac{1}{\tau} \log |\rho_j|$; cf. Colonius and Kliemann [14, Theorem 7.2.9]. In particular, 1 is a Floquet multiplier then 0 is a Floquet exponent.

We also refer to Chicone [11, Section 2.4] and Teschl [41, Section 3.6] for the Floquet theory (note that the Floquet exponents defined in (1.10) are the real parts of the Floquet exponents defined in [11], [25] and [41]).

For the inhomogeneous equation, the following results on periodic solutions are classical (we include a proof for the reader's convenience). Note that here uniqueness of a periodic solution means that it is unique up to time shifts.

Proposition 1.2.1. *Consider the inhomogeneous τ-periodic differential equation (1.8).*

(i) This differential equation has a unique τ-periodic solution if and only if 1 is not an eigenvalue of the matrix $\Phi(\tau, 0)$. The initial value of the unique τ-periodic solution is $x^0 = (I - \Phi(\tau, 0))^{-1} \int_0^\tau \Phi(\tau, s)z(s)ds$.

(ii) If there does not exist a τ-periodic solution of the τ-periodic differential equation (1.8), then the principal matrix solution satisfies 1 is a eigenvalue of the matrix $\Phi(\tau, 0)$ and $\int_0^\tau z(t)dt \notin \operatorname{Im}(I - \Phi(\tau, 0))$.

(iii) For $k = 0, 1, \ldots,$ let $P^k(t)$ and $z^k(t), t \in \mathbb{R}$, be τ_k-periodic with $\tau_k \geq 0$, and assume that the corresponding principal fundamental matrices Φ^k satisfy $1 \notin \operatorname{spec}(\Phi^k(\tau_k, 0))$. Furthermore, suppose that

- *$\tau_k \to \tau_0,\ P^k(\cdot) \to P^0(\cdot)$ in $L^\infty(\mathbb{R}, \mathbb{R}^{n\times n})$;*
- *$z^k(\cdot) \to z^0(\cdot)$ in $L^2([0, \tau_0 + 1]\,; \mathbb{R}^n)$ for $k \to \infty$.*

Then the initial values x^k of the corresponding unique τ_k-periodic solutions converge for $k \to \infty$ to the initial value x^0 of the unique τ_0-periodic solution for $P^0(\cdot)$ and $z^0(\cdot)$.

Proof. (i) A τ-periodic solution $x(\cdot)$ with $x(0) = x^0$ satisfies

$$x^0 = x(\tau) = \Phi(\tau, 0)x^0 + \int_0^\tau \Phi(\tau, s)z(s)ds,$$

and hence

$$(I - \Phi(\tau, 0))x^0 = \int_0^\tau \Phi(\tau, s)z(s)ds. \tag{1.11}$$

If $1 \notin \operatorname{spec}(\Phi(\tau, 0))$ there is a unique solution $x^0 \in \mathbb{R}^n$ of (1.11). If $1 \in \operatorname{spec}(\Phi(\tau, 0))$, i.e., $\ker(I - \Phi(\tau, 0)) \neq \{0\}$, then the inhomogeneous linear equation in $\mathbb{R}^n$ either has no solution or there is a nontrivial affine subspace of solutions and, in particular, the τ-periodic solution is not unique.

(ii) This follows by the arguments in (i).

(iii) Using Gronwall's inequality one shows that for initial values $y^k \to y^0$ the solutions $\psi^k(t, y^k)$ of $\dot{x}(t) = P^k(t)x(t) + z^k(t),\ \psi^k(0, y^k) = y^k$, converge uniformly on bounded intervals to $\psi^0(t, y^0)$. This also holds for the principal fundamental solutions and hence, for $z^k(t) \not\equiv 0$,

$$x^k = (I - \Phi^k(\tau_k, 0))^{-1} \int_0^{\tau_k} \Phi^k(\tau_k, s) z^k(s) ds$$

converges to

$$x^0 = (I - \Phi^0(\tau_0, 0))^{-1} \int_0^{\tau_0} \Phi^0(\tau_0, s) z^0(s) ds.$$

For $z^k(t) \equiv 0$, the convergence trivially holds. □

Suppose that u is a τ-periodic control, i.e., $u(t + \tau) = u(t)$ for all $t \in \mathbb{R}$, then $A(u(t + \tau)) = A(u(t))$ and $C(u(t+\tau)) + d = C(u(t)) + d$, so for periodic control the equation (1.5) is τ-periodic. Therefore, all results on periodic solutions can be applied to affine control systems.

The last result of this section establishes the relationship between the orbits of an element determined by an affine system and the orbits of an element determined by the time reverse system.

Lemma 1.2.2. *Consider affine control system on $\mathbb{R}^n$*

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} v_i(t) f_i(x(t)), \quad v \in \mathcal{U}. \tag{1.12}$$

The time reversed system is the system on $\mathbb{R}^n$ given by

$$\dot{x}(t) = -f_0(x(t)) - \sum_{i=1}^{m} v_i(t) f_i(x(t)), \quad v \in \mathcal{U}. \tag{1.13}$$

We denote by $\mathcal{O}_1^+(x)$ and $\mathcal{O}_1^-(x)$ the reachable set from x and the controllable set to x, determined by the system (1.12), respectively, and by $\mathcal{O}_2^+(x)$ and $\mathcal{O}_2^-(x)$ the reachable set from x and the controllable set to x, determined by the system (1.13), respectively. Then $\mathcal{O}_1^+(x) = \mathcal{O}_2^-(x)$ and $\mathcal{O}_1^-(x) = \mathcal{O}_2^+(x)$.

Proof. For $y = \varphi(T, x, u) \in \mathcal{O}_1^+(x)$, the absolutely continuous function $\psi(t) := \varphi(T - t, x, u(T - \cdot)), t \in [0, T]$, satisfies $\psi(0) = y, \psi(T) = x$. It is a solution of (1.13) with

$v(t) := u(T-t), t \in [0,T]$, since for almost all $t \in [0,T]$

$$\begin{aligned}\dot{\psi}(t) &= \frac{d}{dt}\varphi(T-t,y,u(T-\cdot))\\ &= -f_0(\varphi(T-t,y,u(T-\cdot))) - \sum_{i=1}^{m} u_i(T-t) f_i(\varphi(T-t,y,u(T-\cdot)))\\ &= -f_0(\psi(t)) - \sum_{i=1}^{m} v_i(t) f_i(\psi(t)).\end{aligned}$$

Thus $\mathcal{O}_1^+(x) \subset \mathcal{O}_2^-(x)$. The other inclusions follow analogously. □

1.3 System semigroup of control system on smooth manifold

In this section we consider control systems which generate a finite dimensional Lie group and analyze the interior of the system semigroup. The concepts and results presented in this section are based on Jurdjevic [27] and we will often refer to it.

Consider a control system of the form (1.1) on a real analytic manifold M given by complete real analytic vector fields $f_0, \ldots, f_m$,

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \quad u \in \mathcal{U}_{pc}.$$

Assume that the associated family of vector fields

$$\mathcal{F} = \left\{ f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \mid u \in \Omega \right\}$$

generates a finite dimensional Lie subalgebra $\mathcal{L}(\mathcal{F})$ of the set of real analytic vector fields on M. Let $\mathcal{G}$ the connected and simple connected group associated with $\mathcal{L}(\mathcal{F})$, the Lie algebra of the right-invariant vector fields defined in $\mathcal{G}$ is isomorphic to $\mathcal{L}(\mathcal{F})$, i. e., for every $X \in \mathcal{F}$ there is a unique corresponding right-invariant vector fields X_r, the family of invariant vector field determined by $\mathcal{F}$ is denoted by $\mathcal{F}_r = \{X_r \mid X \in \mathcal{F}\}$.

The elements g of $\mathcal{G}$ have the form

$$g = \exp(t_1 X_1) \cdots \exp(t_k X_k) \text{ for some elements } X_i \in \mathcal{F} \text{ and } t_i \in \mathbb{R}.$$

The group $\mathcal{G}$ coincides with the orbit of the identity for the vector fields $\mathcal{F}_r$. Denote by $\mathcal{S}_\tau = \mathcal{S}_\tau(\mathcal{F})$ the set of those elements of $\mathcal{G}$ with $t_i \geq 0$ and $t_1 + \cdots + t_k = \tau$, and let $\mathcal{S} = \mathcal{S}(\mathcal{F}) = \bigcup_{\tau > 0} \mathcal{S}_\tau$ be the system semigroup. Also denote $\mathcal{S}_{\leq T} = \bigcup_{\tau \in [0,T]} \mathcal{S}_\tau$.

The action of the system group $\mathcal{G}$ on M is determined by the following: for $X(x) =$

$f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \in \mathcal{F}, u \in \Omega$, and $t \in \mathbb{R}$,

$$\exp(tX)x = \varphi(t, x, u).$$

Throughout this section, we assume that the family $\mathcal{F}$ satisfies the rank condition (1.3), by Jurdjevic [27, Theorem 3 on pg.44] the set $\mathcal{F}$ of vector fields is transitive on M, i.e., for all $x, y \in M$ there is $g \in \mathcal{G}$ with $y = gx$. The trajectories of control system (1.1) are given by the action of the semigroup $\mathcal{S}$ on M : For $g \in \mathcal{S}_\tau$ and $\tau = t_k + \cdots + t_1, t_i > 0$,

$$gx = \exp\left(t_k X^{u^k}\right) \cdots \exp\left(t_1 X^{u^1}\right) x = \varphi(\tau, x, u),$$

where $u^j \in \Omega, X^{u^j} = f_0(x) + \sum_{i=1}^{m} u_i^j f_i(x) \in \mathcal{F}$, and $\varphi(t, x, u), t \in [0, \tau]$, is the solution of (1.1) with control defined, with $t_0 = 0$, by

$$u(t) := u^{j+1} \text{ for } t \in \left[\sum_{i=0}^{j} t_i, \sum_{i=0}^{j+1} t_i\right), \quad j = 0, \ldots, k-1.$$

Theorem 1.3.1. *Consider the system semigroup of the system (1.1). Then the following properties are holds:*

(i) The system semigroup $\mathcal{S} = \mathcal{S}(\mathcal{F})$ satisfies $\mathcal{S}_{\leq\tau} \subset \overline{\text{int}(\mathcal{S}_{\leq\tau})}$ in $\mathcal{G}$ for every $\tau > 0$.

(ii) If $g \in \text{int}(\mathcal{S}_{\leq\tau})$ for a $\tau > 0$ then $gx \in \text{int}(\mathcal{O}^+_{\leq\tau}(x))$ for every $x \in M$.

Proof. (i) The invariant vector fields in a Lie group are analytic; cf. Jurdjevic [27, p. 69], this implies that the family $\mathcal{F}_r$ is Lie-determined (satisfy the rank condition (1.3)), so we can apply Corollary in Jurdjevic [27, Corollary on p. 67] which shows that for every open set U in an orbit of $\mathcal{F}_r$, any $y \in U$, and any $T > 0$, the reachable set $\mathcal{S}_{\leq T}(\mathcal{F}_r)(y) \cap U$ contains an open set in the orbit topology. In particular, this applies to the reachable set up to time T of the identity which coincides with $\mathcal{S}_{\leq T}$. Furthermore, [27, Corollary 1 on p. 68] implies $\mathcal{S}_{\leq\tau} \subset \overline{\text{int}(\mathcal{S}_{\leq\tau})}$.

(ii) The maps $\mathcal{G} \longrightarrow M$ defined by $g \longmapsto gx$ are open. If $g \in \text{int}\left(\mathcal{S}_{\leq T}\right)$ then $gx \in \text{int}\left(\mathcal{O}^+_{\leq T}(x)\right)$. □

In the following, we always consider the interior of $\mathcal{S}$ in the system group $\mathcal{G}$. As $\text{int}(\mathcal{S}) = \bigcup_{T>0} \mathcal{S}_{<T}$ then if $g \in \text{int}(\mathcal{S})$ implies that $g \in \text{int}(\mathcal{S}_{\leq\tau})$ for some $\tau > 0$; cf. Colonius and Kliemann [13, Lemma 4.5.2]. Theorem 1.3.1 leads to the following result connecting elements g in the interior of the system semigroup and fixed points of the control system.

Proposition 1.3.2. *(i) Let $D \subset M$ be a control set with nonvoid interior. Then for every $x \in \text{int}(D)$ there are $\tau > 0$ and $g \in \mathcal{S}_\tau \cap \text{int}\left(\mathcal{S}_{\leq\tau+1}\right)$ such that $gx = x$.*

(ii) Conversely, let $g \in \text{int}\left(\mathcal{S}_{\leq\tau}\right)$ for some $\tau > 0$ with $gx = x$ for some point $x \in M$. Then $x \in \text{int}(D)$ for some control set $D \subset M$.

Proof. (i) Let $x \in \operatorname{int}(D)$. By continuity of the action we have that the set $H = \{h \in \mathcal{G} \mid hx \in \operatorname{int}(D)\}$ is open in $\mathcal{G}$. Since D is a control set, there exists $u \in \mathcal{U}_{pc}$ such that $\varphi(t,x,u) \in D$ for all $t \geq 0$, that is, for every $t \geq 0$ there is $h_t \in \mathcal{G}$ with $\varphi(t,x,u) = h_t x \in D$, so $h_t \in \mathcal{S}_t \cap H$. By normal accessibility; cf. Jurdjevic [27, Theorem 1 on p. 66] for any neighborhood V of x there is a state $y \in V$ such that $y \in \mathcal{O}^+_{\leq T}(x)$, for some $T > 0$, as consequence $V \cap \mathcal{O}^+_{\leq T}(x)$ contain a open subset. There are for $\sigma_k \to 0^+$ elements $g_k \in \operatorname{int}(\mathcal{S}_{\leq \sigma_k})$ converging to the identity in $\mathcal{G}$, and hence $h_k := g_k h \in \operatorname{int}(\mathcal{S}_{\leq t+\sigma_k}) \to h$. Since H is open we can fix $k \in \mathbb{N}$ such that $h_k \in H \cap \mathcal{S}_{t+\sigma} \cap \operatorname{int}(\mathcal{S}_{\leq t+\sigma_k})$ for some $\sigma \in [0, \sigma_k]$, hence $h_k x \in \operatorname{int}(D)$. By exact controllability in $\operatorname{int}(D)$, we find $h_0 \in \mathcal{S}_s$, $s > 0$, such that $h_0 h_k x = x$. It follows that $g := h_0 h_k \in \mathcal{S}_\tau \cap \operatorname{int}(\mathcal{S}_{\leq \tau+1})$ with $\tau := t + \sigma + s$ and $gx = x$.

(ii) Every $g \in \operatorname{int}(\mathcal{S}_{\leq \tau})$ satisfies $gx \in \operatorname{int}(\mathcal{O}^+(x))$ and $x \in \operatorname{int}(\mathcal{O}^-(gx))$ for all x. Now $gx = x$ implies that

$$x \in \operatorname{int}\left(\mathcal{O}^+(x)\right) \cap \operatorname{int}\left(\mathcal{O}^-(x)\right),$$

and hence Lemma 1.1.9 shows that $D = \mathcal{O}^-(x) \cap \overline{\mathcal{O}^+(x)}$ is a control set with $x \in \operatorname{int}(D)$. □

Next, we will prove that the interior of the system semigroup is path-connected and for that we will need this lemma.

Lemma 1.3.3. *Les $\mathcal{S}$ be the system semigroup of the system (1.1). If $g' \in \mathcal{S}$ and $g'' \in \operatorname{int}(\mathcal{S})$. Then $g'g'', g''g' \in \operatorname{int}(\mathcal{S})$.*

Proof. This follows from the fact that the translations in $\mathcal{G}$ are diffeomorphisms and so $g'\operatorname{int}(\mathcal{S})$ and $\operatorname{int}(\mathcal{S})g'$ are open. □

The following lemma shows that the interior of the system semigroup is path-connected such that the corresponding controls and times change continuously along of the paths.

Lemma 1.3.4. *Let $g \in \mathcal{S}_\sigma \cap \operatorname{int}(\mathcal{S})$ and $h \in \mathcal{S}_\tau \cap \operatorname{int}(\mathcal{S})$ for some $\sigma, \tau > 0$. Then there is a continuous path $p : [\sigma, 2\sigma+\tau] \to [0,\infty) \times \operatorname{int}(\mathcal{S})$ with $p(\sigma) = (\sigma, g)$ and $p(2\sigma+\tau) = (\tau, h)$ such that*

$$p(\alpha) = \begin{cases} (\alpha, g_\alpha) & \text{for } \alpha \in [\sigma, \sigma+\tau] \\ (2\sigma + 2\tau - \alpha, g_{2\sigma+2\tau-\alpha}) & \text{for } \alpha \in (\sigma+\tau, 2\sigma+\tau] \end{cases}$$

with $g_\alpha \in \mathcal{S}_\alpha \cap \operatorname{int}(\mathcal{S})$ for $\alpha \in [\sigma, \sigma+\tau]$ and $g_{2\sigma+2\tau-\alpha} \in \mathcal{S}_{2\sigma+2\tau-\alpha} \cap \operatorname{int}(\mathcal{S})$ for $\alpha \in (\sigma+\tau, 2\sigma+\tau]$.

Proof. We may write,

$$g = \exp(s_k X_k) \cdots \exp(s_1 X_1), \quad h = \exp(t_\ell Y_\ell) \cdots \exp(t_1 Y_1), \tag{1.14}$$

where $k, \ell \in \mathbb{N}$, $t_i, s_i > 0$, and $X_i, Y_i \in \mathcal{F}$ for all i. Thus $\sigma = s_k + \cdots + s_1$ and $\tau = t_\ell + \cdots + t_1$. First define a continuous path

$$\begin{array}{rccc} p: & [\sigma, \sigma+\tau] & \longrightarrow & [\sigma, \sigma+\tau] \times \operatorname{int}(\mathcal{S}) \\ & \alpha & \longmapsto & (\alpha, g_\alpha) \end{array}$$

where g_α is defined by: Set $t_0 = 0$, for $j = 1, \ldots, \ell$ and $\alpha \in \left[\sigma + \sum_{i=0}^{j-1} t_i, \sigma + \sum_{i=0}^{j} t_i\right)$,

$$g_\alpha = \exp\left(\left(\alpha - \sigma - \sum_{i=0}^{j-1} t_i\right) Y_j\right) \exp\left(t_{j-1} Y_{j-1}\right) \cdots \exp\left(t_1 Y_1\right) g.$$

By [33, Theorem 18.3 on pg.108] the function p is continuous. Then $g_\sigma = g$, $g_{\sigma+\tau} = hg$ and $g_\alpha \in \mathcal{S}_\alpha \cap \operatorname{int}(\mathcal{S})$, by Lemma 1.3.3.

Analogously, we can connect $(\sigma+\tau, hg)$ continuously with (τ, h) implying the assertion.

□

Remark 1.3.5. *The elements g_α constructed in the proof above correspond to control functions u^α given by the following: let $g = g(u)$ and $h = h(v)$ with*

$$v(t) = v^j \text{ for } t \in \left[\sum_{i=0}^{j-1} t_i, \sum_{i=0}^{j} t_i\right), \quad j = 1, \ldots, \ell,$$

where v^j determines the vector field Y_j. Then for $\alpha \in \left[\sigma + \sum_{i=0}^{j-1} t_i, \sigma + \sum_{i=0}^{j} t_i\right)$, $j = 1, \ldots, \ell$, the element g_α is determined by the control

$$u^\alpha(t) = \begin{cases} u(t) & \text{for } t \in [0, \sigma] \\ v^j & \text{for } t \in \left[\sigma + \sum_{i=0}^{j-1} t_i, \alpha\right), \quad j = 1, \ldots, \ell \end{cases}$$

and analogously for $\alpha \in [\sigma + \tau, 2\sigma + \tau]$. It follows that τ_α and g_α depend continuously on α and τ_α depends in a piecewise analytic way on α. Furthermore, also the controls $u^\alpha \in L^2([0, \sigma+\tau]; \mathbb{R}^m)$ depend continuously on α.

1.4 Systems semigroup of control system on $\mathbb{R}^n$

In this section we analyze the system semigroup specifically for control system (1.5) on $\mathbb{R}^n$.

The family of vector fields on $\mathbb{R}^n$ associated with (1.5) is given by

$$\mathcal{F} = \left\{ X^u(x) = \left(A + \sum_{i=1}^m u_i B_i\right) x + \sum_{i=1}^m u_i c_i + d \;\middle|\; u \in \Omega \right\}.$$

The system group $\mathcal{G} = \mathcal{G}(\mathcal{F})$ is a subgroup of the semidirect product $\mathbb{R}^n \ltimes GL(\mathbb{R}^n)$; cf. Jurdjevic and Sallet [29]. For an affine vector field $X(x) = A_0 x + a_0 \in \mathcal{F}$ we write $\exp tX$

for the one parameter group of diffeomorphisms generated by X. Then, for $x \in \mathbb{R}^n$,

$$(\exp tX)x = e^{tA_0}x + \int_0^t e^{(t-s)A_0}a_0\, ds.$$

Next we describe the relation between the action of the system semigroup and periodic control functions. For $g \in \mathcal{S}_\tau$ and $\tau = t_k + \cdots + t_1$,

$$gx = \exp\left(t_k X^{u^k}\right) \cdots \exp\left(t_1 X^{u^1}\right) x = \varphi(\tau, x, u), \tag{1.15}$$

where $u^j \in \Omega$ and X^{u^j} is the vector field

$$X^{u^j}(x) = \left(A + \sum_{i=1}^m u_i^j B_i\right) x + d + \sum_{i=1}^m u_i^j c_i,$$

and $\varphi(t, x, u), t \in [0, \tau]$, is the solution of (1.5) with control defined by

$$u(t) := u^{j+1} \text{ for } t \in \left[\sum_{i=0}^j t_i, \sum_{i=0}^{j+1} t_i\right) \text{ and } j = 0, \ldots, k-1. \tag{1.16}$$

We may consider u as a τ-periodic control function when we extend it τ-periodically to $\mathbb{R}$. Since an element $g \in \mathcal{S}_\tau$ is determined by a τ-periodic control u, we may write it as $g(u)$. The principal fundamental solution of $\dot{x}(t) = A(u(t))x(t)$ corresponding to u is $\Phi_u(t, s)$. Observe that u also determines an element of the system semigroup of this bilinear homogeneous system.

We note the following simple but important lemma.

Lemma 1.4.1. *Let $g(u) \in \mathcal{S}_\tau$. Then $g(u)x = x$ for some $x \in \mathbb{R}^n$ if and only if the inhomogeneous τ-periodic differential equation*

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^m u_i(t)(B_i x(t) + c_i) + d = A(u(t))x(t) + Cu + d$$

has a τ-periodic solution with initial value $x(0) = x$. This solution is unique if and only if $1 \notin \operatorname{spec}(\Phi_u(\tau, 0))$.

Proof. The first assertion is clear by the definitions. The second assertion follows from Proposition 1.2.1(i). □

Next we use Lemma 1.3.4 for a result on spectral properties for elements in the interior of the system semigroup.

Lemma 1.4.2. *Let $g(u^0) \in \mathcal{S}_\sigma \cap \operatorname{int}(\mathcal{S})$ for some $\sigma > 0$ with $1 \notin \operatorname{spec}(\Phi_{u^0}(\sigma, 0))$ and suppose that there is $g(v) \in \mathcal{S}_\tau \cap \operatorname{int}(\mathcal{S})$ with $\tau > 0$ and $1 \in \operatorname{spec}(\Phi_v(\tau, 0))$.*

Then there exist $\tau_1 > 0$, $g(u^1) \in \mathcal{S}_{\tau_1} \cap \operatorname{int}(\mathcal{S})$ and a continuous path

$$\begin{array}{rccc} p: & [0,1] & \longrightarrow & \operatorname{int}(\mathcal{S}) \\ & \alpha & \longmapsto & p(\alpha) = g(u^\alpha) \end{array}$$

where the u^α are τ_α-periodic controls with $g(u^\alpha) \in \mathcal{S}_{\tau_\alpha} \cap \operatorname{int}(\mathcal{S})$ for $\alpha \in [0,1]$, $p(0) = g(u^0)$ and $p(1) = g(u^1)$ and the following properties hold:

The principal fundamental matrix $\Phi^\alpha(t,s)$ of $\dot{x}(t) = A(u^\alpha(t))x(t)$ satisfy :

(i) $1 \notin \operatorname{spec}(\Phi^\alpha(\tau_\alpha, 0))$ *for* $\alpha \in [0,1)$;

(ii) and $1 \in \operatorname{spec}(\Phi_{u^1}(\tau_1, 0))$.

The matrices $\Phi^\alpha(t,s)$, the periods τ_α as well as the controls $u^\alpha \in L^2(0, \sigma + \tau^1; \mathbb{R}^m)$ depend continuously on α. The numbers τ_α as well as the matrices $\Phi_{u^\alpha}(\tau_\alpha, 0)$ depend in a piecewise analytic way on α.

Proof. By Lemma 1.3.4 there is a continuous path $p_1 : [\sigma, 2\sigma+\tau] \to [0,\infty) \times \operatorname{int}(\mathcal{S})$ with $p_1(\sigma) = (\sigma, g(u^0))$ and $p_1(2\sigma+\tau) = (\tau, g(v))$, such that

$$p_1(\alpha) = \begin{cases} (\alpha, g_\alpha) & \text{for} \quad \alpha \in [\sigma, \sigma+\tau] \\ (2\sigma + 2\tau - \alpha, g_{2\sigma+2\tau-\alpha}) & \text{for} \quad \alpha \in (\sigma+\tau, 2\sigma+\tau] \end{cases}$$

with $g_\alpha \in \mathcal{S}_\alpha \cap \operatorname{int}(\mathcal{S})$ for $\alpha \in [\sigma, \sigma+\tau]$ and $g_{2\sigma+2\tau-\alpha} \in \mathcal{S}_{2\sigma+2\tau-\alpha} \cap \operatorname{int}(\mathcal{S})$ for $\alpha \in (\sigma+\tau, 2\sigma+\tau]$. The construction in the proof of Lemma 1.3.4 shows that $g_\alpha = g(u^\alpha)$ with controls u^α depending continuously on α as elements in $L^2([0,\sigma+\tau]; \mathbb{R}^m)$. Hence also

$$A(u^\alpha(\cdot)) = A + \sum_{i=1}^m u_i^\alpha(\cdot) B_i \in L^2(0, \sigma+\tau; \mathbb{R}^{n\times n})$$

depends continuously on α. The principal fundamental matrices $\Phi^\alpha(t,s)$ are given by

$$\Phi^\alpha(t,s) = I + \int_0^s A(u^\alpha(s'))\Phi^\alpha(t,s')ds'.$$

Gronwall's lemma implies that these matrices also depend continuously on α, uniformly in t and s on bounded intervals, since

$$\begin{aligned} & \Phi^\alpha(t,s) - \Phi^\beta(t,s) \\ = & \int_0^s \left[A(u^\alpha(s'))\Phi^\alpha(t,s') - A(u^\beta(s'))\Phi^\beta(t,s')\right] ds' \\ = & \int_0^s \left[A(u^\alpha(s')) - A(u^\beta(s'))\right] \Phi^\alpha(t,s')ds' \\ & + \int_0^s A(u^\beta(s')) \left[\Phi^\alpha(t,s')ds' - \Phi^\beta(t,s')\right] ds', \end{aligned}$$

implying

$$\begin{aligned}\left\|\Phi^\alpha(t,s)-\Phi^\beta(t,s)\right\| &\leq \int_0^s \left\|A(u^\alpha(s'))-A(u^\beta(s'))\right\|^2 ds' \int_0^s \left\|\Phi^\alpha(t,s')\right\|^2 ds' \\ &\quad + t \max_{s'\in[0,s]} \left\|A(u^\beta(s'))\right\| \max_{s'\in[0,s]} \left\|\Phi^\alpha(t,s')ds' - \Phi^\beta(t,s')\right\|.\end{aligned}$$

Thus also the eigenvalues of $\Phi^\alpha(t,0)$ depend continuously on α. Then one of the following two cases occurs:

(i) There is $\alpha_0 \in [\sigma,\sigma+\tau]$ such that $1 \in \mathrm{spec}(\Phi^{\alpha_0}(\alpha_0,0))$ and $1 \notin \mathrm{spec}(\Phi^\alpha(\alpha,0))$ for all $\alpha \in [0,\alpha_0)$. It follows that for every $\alpha \in [0,\alpha_0)$ there is a unique α-periodic solution of the corresponding equation in (1.5).

(ii) There is $\alpha_0 \in [\sigma+\tau, 2\sigma+\tau)$ such that $1 \in \mathrm{spec}(\Phi^{2\sigma+2\tau-\alpha_0}(2\sigma+2\tau-\alpha_0,0))$ and

$$\begin{aligned}&1 \notin \mathrm{spec}(\Phi^\alpha(\alpha,0)) \text{ for all } \alpha \in [0,\sigma+\tau] \text{ and}\\ &1 \notin \mathrm{spec}(\Phi^{2\sigma+2\tau-\alpha}(2\sigma+2\tau-\alpha,0)) \text{ for all } \alpha \in [\sigma+\tau,\alpha_0).\end{aligned}$$

Here one finds as in case (i) unique periodic solutions.

Reparametrizing the path so that $p(1) = g(u^{\alpha_0})$ one obtains the continuous path p. Finally, Remark 1.3.5 shows that in the intervals where the control u^α is constant, it depends in an analytic way on α. This also follows for the periods τ_α and the principal fundamental matrix $\Phi^\alpha(\tau_\alpha,0)$. □

Lemma 1.4.3. *Consider, for $k = 0,1,\ldots,$ the differential equations*

$$\dot{x}(t) = A\left(u^k(t)\right)x(t) + Cu^k(t) + d$$

and suppose that

(i) the control u^k is τ^k-periodic with $\tau^k \to \tau^0 > 0$ and $u^k \to u^0$ in $L^2([0,\tau^0+1];\mathbb{R}^m)$;

(ii) the principal fundamental solutions of $\dot{x}(t) = A(u^k(t))x(t)$ satisfy $1 \in \mathrm{spec}(\Phi^0(\tau^0,0))$ and $1 \notin \mathrm{spec}(\Phi^k(\tau^k,0))$ for $k = 1,2,\ldots$;

(iii) $\int_0^{\tau^0} \Phi^0(\tau^0,s)\left(Cu^0(s)+d\right)ds \notin \mathrm{Im}(I-\Phi^0(\tau^0,0))$.

Then, for $k = 1,2,\ldots$ the initial values x^k of the unique τ^k-periodic solutions determined by

$$(I-\Phi^k(\tau^k,0))x^k = \int_0^{\tau^k} \Phi^k(\tau^k,s)\left(Cu^k(s)+d\right)ds,$$

satisfy for $k \to \infty$

$$\left\|x^k\right\| \to \infty \text{ and } \frac{x^k}{\|x^k\|} \to \ker(I-\Phi^0(\tau^0,0)) = \mathbf{E}(\Phi^0(\tau^0,0);1). \tag{1.17}$$

Proof. As in Lemma 1.4.2, one shows that the principal fundamental solutions $\Phi^k(t,s)$ of $\dot{x}(t) = A(u^k(t))x(t)$ satisfy $\Phi^k(\tau^k, 0) \to \Phi^0(\tau^0, 0)$ for $k \to \infty$. Assume, by way of contradiction, that x^k remains bounded, hence we may suppose that there is $x^0 \in \mathbb{R}^n$ with $x^k \to x^0$, if necessary we consider a subsequence. Then it follows from $u^k \to u^0$ in $L^2([0, \tau^0 + 1]; \mathbb{R}^m)$ that

$$(I - \Phi^0(\tau^0, 0))x^0 = \int_0^{\tau^0} \Phi^0(\tau^0, s) \left(Cu^0(s) + d\right) ds,$$

contradicting assumption (iii). We have shown that x^k becomes unbounded for $k \to \infty$. Since $Cu^k(\cdot) + d$ remains bounded in $L^2([0, \tau^0 + 1]; \mathbb{R}^n)$, we get

$$(I - \Phi^k(\tau^k, 0))\frac{x^k}{\|x^k\|} = \frac{1}{\|x^k\|} \int_0^{\tau^k} \Phi^k(\tau^k, s) \left(Cu^k(s) + d\right) ds \to 0.$$

Then (1.17) follows. □

The next lemma describes the case where assumption (iii) above not valid.

Lemma 1.4.4. *Consider for a τ_0-periodic control u^0*

$$\dot{x}(t) = A(u^0(t))x(t) + Cu^0(t) + d, \tag{1.18}$$

and suppose that the principal fundamental matrix $\Phi_{u^0}(t,s)$ satisfy $1 \in \operatorname{spec}(\Phi(\tau_0, 0))$ and for some $y^0 \in \mathbb{R}^n$

$$\int_0^{\tau^0} \Phi_{u^0}(\tau_0, s) \left(Cu^0(s) + d\right) ds = (I - \Phi_{u^0}(\tau_0, 0))y^0.$$

Then the nontrivial affine subspace $Y = y^0 + \ker(I - \Phi_{u^0}(\tau_0, 0)) = y^0 + \mathbf{E}(\Phi_{u^0}(\tau_0, 0); 1)$ has the property that there is a τ_0-periodic solution of (1.18) starting in y if and only if $y \in Y$, and there are $x^k \in Y, k \in \mathbb{N}$, satisfying for $k \to \infty$ the conditions in (1.17).

Proof. Let $y \in Y$ be so $y = y^0 + z$ with $z \in \ker(I - \Phi_{u^0}(\tau_0, 0))$. We have

$$\begin{aligned}
y &= \Phi_{u^0}(\tau_0, 0)y^0 + \int_0^{\tau_0} \Phi_{u^0}(\tau_0, s) \left(Cu^0(s) + d\right) ds + z \\
&= \Phi_{u^0}(\tau_0, 0)y^0 + \int_0^{\tau_0} \Phi_{u^0}(\tau_0, s) \left(Cu^0(s) + d\right) ds + \Phi_{u^0}(\tau_0, 0)z \\
&= \Phi_{u^0}(\tau_0, 0)(y^0 + z) + \int_0^{\tau_0} \Phi_{u^0}(\tau_0, s) \left(Cu^0(s) + d\right) ds \\
&= \Phi_{u^0}(\tau_0, 0)y + \int_0^{\tau_0} \Phi_{u^0}(\tau_0, s) \left(Cu^0(s) + d\right) ds.
\end{aligned}$$

Thus there is a τ_0-periodic solution of (1.18) starting in y. On the other hand, if there is a τ_0-periodic solution of (1.18) starting in y then

$$\begin{aligned} y &= \Phi_{u^0}(\tau_0, 0)y + \int_0^{\tau_0} \Phi_{u^0}(\tau_0, s)\big(Cu^0(s) + d\big)\, ds \\ &= \Phi_{u^0}(\tau_0, 0)y + (I - \Phi_{u^0}(\tau_0, 0))y^0 \\ &= y^0 + \Phi_{u^0}(\tau_0, 0)(y - y_0), \end{aligned}$$

that is, $y \in Y$.

For the second assertion choosing $x^k := y^0 + kz$, $k \in \mathbb{N}$, where $0 \neq z \in \mathbf{E}(\Phi_{u^0}(\tau_0, 0); 1)$. Then x^k satisfies $\|y^0 + kz\| \to \infty$ for $k \to \infty$ and

$$\frac{x^k}{\|x^k\|} = \frac{y^0 + kz}{\|y^0 + kz\|} \to \frac{z}{\|z\|} \in \mathbf{E}(\Phi_{u^0}(\tau^0, 0); 1).$$

□

Chapter 2

Control sets for homogeneous bilinear systems

In this chapter we cite several results on control sets and chain control sets for homogeneous bilinear control systems defined on $\mathbb{R}^n$, many of these results were obtained from results valid for bilinear systems defined in the unit sphere $\mathbb{S}^{n-1}$ or in the projective space $\mathbb{P}^{n-1}$. Section 2.1 we recall the definition of semitopologically conjugated systems, and associate a bilinear system in $\mathbb{R}^n$ with bilinear systems in the sphere and in the projective space. Section 2.2 we present the results on control sets for bilinear systems in $\mathbb{R}^n$, analyze the relations between the control sets for bilinear systems on $\mathbb{R}^n$, and control sets for the projected systems on the unit sphere $\mathbb{S}^{n-1}$ and also we analyze the relations between the control sets for the projected systems on projective space $\mathbb{P}^{n-1}$ and the system on the unit sphere $\mathbb{S}^{n-1}$.

2.1 Projected system

The following process will be very useful for our work: From an homogeneous bilinear control system in $\mathbb{R}^n$ we associate a control system in the unit sphere $\mathbb{S}^{n-1}$, or in projective space $\mathbb{P}^{n-1}$, so that properties of the homogeneous bilinear control system can be obtained from these, so we consider the following definition.

The results of this section can be found in Kawan [30], and presented here for ease of understanding.

Definition 2.1.1. *Consider two control systems*

$$\dot{x}(t) = F(x(t), u(t)),\ \ u \in \mathcal{U} \tag{2.1}$$

and

$$\dot{y}(t) = G(y(t), v(t))\ \ v \in \mathcal{V} \tag{2.2}$$

on smooth manifolds M and N, with trajectories φ and ψ and with control ranges U and V, respectively, where F and G are complete vector fields and $\mathcal{U}$ and $\mathcal{V}$ denote the corresponding families of admissible control functions. Let $\pi : M \longrightarrow N$ and $h : \mathcal{U} \longrightarrow \mathcal{V}$ be continuous maps such that the following identity is holds:

$$\pi(\varphi(t,x,u)) = \psi(t,\pi(x),h(u)),\ \forall\ (t,x,u) \in \mathbb{R} \times M \times \mathcal{U}.$$

Then we say that the system (2.1) is ***topologically semiconjugate*** *to system (2.2), and we call the pair (π, h) a topological semiconjugacy. If π is a homeomorphism and h is homeomorphism, then the systems are called* ***topologically conjugate*** *and (π, h) is called a topological conjugacy from system (2.1) to system (2.2).*

Given a bilinear system defined on $\mathbb{R}^n$ we will consider topologically conjugated systems defined in projective space or in unit sphere, so we recall here their differentiable structures.

Recall that $\mathbb{P}^{n-1} = (\mathbb{R}^n \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation $x \sim y$ if $y = \lambda x$ for some $\lambda \neq 0$. Furthermore, an atlas of $\mathbb{P}^{n-1}$ is given by n charts (U_i, ψ_i), where U_i is the set of equivalence classes $[x_1 : \cdots : x_n]$ with $x_i \neq 0$ (the homogeneous coordinates) and $\psi_i : U_i \to \mathbb{R}^{n-1}$ is defined by

$$\psi_i([x_1 : \cdots : x_n]) = \left(\frac{x_1}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}\right),$$

where the hat means that the i-th entry is missing.

The unit sphere is $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ an atlas of $\mathbb{S}^{n-1}$ is given by $2n$ charts $(U_i^\pm, \psi_i^\pm)$, where $U_i^+ = \{(x_1, \ldots, x_n) \in \mathbb{S}^{n-1} \mid x_i > 0\}$, $U_i^- = \{(x_1, \ldots, x_n) \in \mathbb{S}^{n-1} \mid x_i < 0\}$ and $\psi_i^\pm : U_i^\pm \to \mathbb{R}^n$ is defined by

$$\psi_i^\pm((x_1, \ldots, x_{n+1})) = (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}),$$

where the hat means that the i-th entry is missing.

Consider a bilinear control system on $\mathbb{R}^n$

$$\dot{x}(t) = A(u)x(t), \quad u \in \mathcal{U}_{pc}. \tag{2.3}$$

We regard (2.3) as a control system on $\mathbb{R}^n \setminus \{0\}$. Consider the radial projection of $\mathbb{R}^n \setminus \{0\}$ onto the sphere $\mathbb{S}^{n-1}$ given by

$$\begin{array}{rccc} \pi_{\mathbb{S}}: & \mathbb{R}^n \setminus \{0\} & \longrightarrow & \mathbb{S}^{n-1} \\ & x & \longmapsto & \pi_{\mathbb{S}}(x) = \frac{x}{\|x\|} \end{array}.$$

It is possible to prove that is $\pi_{\mathbb{S}}$ a smooth submersion with derivative

$$D\pi_{\mathbb{S}}(x) = \frac{1}{\|x\|}\left(I - \frac{xx^\top}{\|x\|^2}\right).$$

The bilinear system (2.3) is topologically semiconjugate to the follow system on $\mathbb{S}^{n-1}$

$$\dot{s} = G(s,u) = (A(u) - s^\top A(u)sI)s, \tag{2.4}$$

which follows from

$$\begin{aligned}
D\pi_{\mathbb{S}}(x)(A(u)x) &= \frac{1}{\|x\|}\left(I - \frac{xx^\top}{\|x\|^2}\right)A(u)x \\
&= \left(A(u) - \frac{xx^\top A(u)}{\|x\|^2}\right)\frac{x}{\|x\|} \\
&= \left(A(u)x - \frac{x(x^\top A(u)x)}{\|x\|^2}\right)\frac{1}{\|x\|} \\
&= \left(A(u) - \frac{x^\top A(u)x}{\|x\|^2}I\right)\frac{x}{\|x\|} \\
&= \big(A(u) - \pi_{\mathbb{S}}(x)^\top A(u)\pi_{\mathbb{S}}(x)I\big)\,\pi_{\mathbb{S}}(x) \\
&= G(\pi_{\mathbb{S}}(x), u).
\end{aligned}$$

In this case, we say that the system (2.3) is projected on the $\mathbb{S}^{n-1}$ and (2.4) is the **projected system** on $\mathbb{S}^{n-1}$.

We also obtain a homogeneous bilinear control system on projective space $\mathbb{P}^{n-1}$ topologically semiconjugate to (2.3), considering the projection of $\mathbb{R}^n\setminus\{0\}$ onto the $\mathbb{P}^{n-1}$ given by

$$\begin{array}{rccc}
\pi_{\mathbb{P}}: & \mathbb{R}^n\setminus\{0\} & \longrightarrow & \mathbb{P}^{n-1} \\
 & x & \longmapsto & \pi_{\mathbb{P}}(x) = [x]
\end{array}$$

where $[x] = \{y \in \mathbb{R}^n\setminus\{0\} \mid y = \lambda x, \text{ for some } \lambda \in \mathbb{R}\setminus\{0\}\}$.

Since $G(-s,u) = (A(u)-(-s)^\top A(u)(-s)I)(-s) = -\big(A(u) - s^\top A(u)sI\big)\,s = -G(s,u)$ and $[d\pi_s] = [d\pi_{-s}]$ we have

$$d\pi_{-s}(G(-s,u)) = d\pi_{-s}(-G(s,u)) = -d\pi_{-s}(G(s,u)) = d\pi_s(G(s,u)).$$

As the projection satisfies $\pi_{\mathbb{P}}(x) = \pi_{\mathbb{P}}(y)$, for $x, y \in \mathbb{S}^{n-1}$, if and only if $x = -y$, so the vector field $H([x],u) = d\pi_{\mathbb{P}[x]}(G(x,u))$ is well defined and the system

$$\dot{[x]} = H([x],u) = d\pi_{\mathbb{P}[x]}(G(x,u)) \tag{2.5}$$

is topologically semiconjugate to (2.4) and also to (2.3). The system (2.5) is the projected system on $\mathbb{P}^{n-1}$.

The solution of the systems (2.4) and (2.5) are denoted by $\varphi_{\mathbb{S}}(t, s, u)$ and $\varphi_{\mathbb{P}}(t, [s], u)$, respectively, and their positive orbits by ${}_{\mathbb{S}}\mathcal{O}^+(s)$ and ${}_{\mathbb{P}}\mathcal{O}^+([s])$, respectively, similarly to negative orbit.

Since bilinear control systems as well as their projections to $\mathbb{S}^{n-1}$ and $\mathbb{P}^{n-1}$ are analytic, for these systems, by Lemma 1.1.5, local accessibility is equivalent to the corresponding accessibility rank condition; cf. Sontag [39, Theorem 12 on p. 179].

2.2 Control sets for bilinear homogeneous system on $\mathbb{R}^n$

In this section we present some results about control sets for homogeneous bilinear systems in $\mathbb{R}^n$, we relate control sets with nonvoid interior of the projected system to control sets for homogeneous bilinear systems in $\mathbb{R}^n$.

Consider the construction and notation given in Section 2.1.

We note the following result showing a first relation between control sets for homogeneous bilinear control system on $\mathbb{R}^n$ and control sets of the projected system. This is a general result, for semi-conjugacy, but we chose to demonstrate it here in our context to make it easier to understand.

Proposition 2.2.1. *Suppose that $D \subset \mathbb{R}^n \setminus \{0\}$ is a control set of system (2.3). Then the projection $\pi_{\mathbb{P}}(D)$ to projective space $\mathbb{P}^{n-1}$ is contained in a control set ${}_{\mathbb{P}}D$ for the projected system (2.5) on $\mathbb{P}^{n-1}$, and the projection $\pi_{\mathbb{S}}(D)$ to the unit sphere $\mathbb{S}^{n-1}$ is contained in a control set ${}_{\mathbb{S}}D$ for projected system (2.4) on $\mathbb{S}^{n-1}$. If D has nonvoid interior, then also ${}_{\mathbb{P}}D$ and ${}_{\mathbb{S}}D$ have nonvoid interiors.*

Proof. Let D be a control set of the system (2.3) and $\pi_{\mathbb{S}}(x) \in \pi_{\mathbb{S}}(D)$. As D is a control set there is $u \in \mathcal{U}_{pc}$ such that $\varphi(t, x, u) \in D$ for all $t \geq 0$, since $\pi_{\mathbb{S}}(\varphi(t, x, u)) = \varphi_{\mathbb{S}}(t, \pi_{\mathbb{S}}(x), u)$, so the solution $\varphi_{\mathbb{S}}(t, \pi(x), u) \in \pi_{\mathbb{S}}(D)$ for all $t \geq 0$. If $y = \varphi(t, x, u)$ for some $u \in \mathcal{U}_{pc}$ and $t \geq 0$, then $\pi_{\mathbb{S}}(y) = \varphi_{\mathbb{S}}(t, \pi(x), u)$, thus $\pi_{\mathbb{S}}(\mathcal{O}^+(x)) \subset {}_{\mathbb{S}}\mathcal{O}^+(\pi_{\mathbb{S}}(x))$. As $\pi_{\mathbb{S}}$ is continuous $\pi_{\mathbb{S}}\left(\overline{\mathcal{O}^+(x)}\right) \subset \overline{{}_{\mathbb{S}}\mathcal{O}^+(\pi_{\mathbb{S}}(x))}$, so $\pi_{\mathbb{S}}(D) \subset \overline{{}_{\mathbb{S}}\mathcal{O}^+(\pi_{\mathbb{S}}(x))}$. Therefore, $\pi_{\mathbb{S}}(D)$ is contained in some control set of projected system (2.4).

For the case of $\pi_{\mathbb{P}}$ is analogous. Since the projections $\pi_{\mathbb{S}}$ and $\pi_{\mathbb{P}}$ are open the second statement follows. $\square$

Viewed from another angle, this proposition shows that each control set of homogeneous bilinear system (2.3) is contained in a cone of the form $\{\alpha s \in \mathbb{R}^n \mid \alpha > 0, s \in {}_{\mathbb{S}}D\}$, where ${}_{\mathbb{S}}D$ is a control set of projected system (2.4). Now we will see some hypotheses that guarantee that these cones are control sets for the homogeneous bilinear system. This result is based on a Diophantine approximation result used for Lemma 2.2.3.

Theorem 2.2.2. *Let ${}_{\mathbb{S}}D$ be a control set with nonvoid interior for the projected system (2.4) on the unit sphere $\mathbb{S}^{n-1}$ and suppose that* *(i) every point in* $\operatorname{int}({}_{\mathbb{S}}D)$ *is locally accessible;*

(ii) there are $\alpha_0^+ > 1,\ \delta_0 > 0,$ *and* $\alpha^- \in (0,1)$*, such that for all* $\alpha^+ \in (\alpha_0^+, \alpha_0^+ + \delta_0)$ *there are points* $s^+, s^- \in \operatorname{int}({}_{\mathbb{S}}D)$*, controls* $u^+, u^- \in \mathcal{U}$*, and times* $\sigma^+, \sigma^- > 0$ *with*

$$\varphi(\sigma^+, s^+, u^+) = \alpha^+ s^+, \quad \varphi(\sigma^-, s^-, u^-) = \alpha^- s^-. \tag{2.6}$$

Then the cone $\{\alpha s \in \mathbb{R}^n \mid \alpha > 0, s \in {}_{\mathbb{S}}D\}$ *is a control set of the homogeneous bilinear system (2.3) on* $\mathbb{R}^n$ *with nonvoid interior.*

The proof of Theorem 2.2.2 will show that we can replace assumption (ii) by the following assumption:

(ii)' there are $\alpha^+ > 1, \delta_0 \in (0,1)$, and $\alpha_0^- \in (0, 1-\delta_0)$ such that for all $\alpha^- \in (\alpha_0^-, \alpha_0^- + \delta_0)$ there are points $s^+, s^- \in \operatorname{int}({}_{\mathbb{S}}D)$, controls $u^+, u^- \in \mathcal{U}$, and times $\sigma^+, \sigma^- > 0$ satisfying (2.6) .

Now, let's go through the proof of the Theorem 2.2.2.

Proof. Considering (2.6) the Property 1.1.6 (iv) implies for the projected system on $\mathbb{S}^{n-1}$

$$\begin{aligned} \varphi_{\mathbb{S}}\left(\sigma^+, s^+, u^+\right) &= s^+, \\ \varphi_{\mathbb{S}}\left(\sigma^-, s^-, u^-\right) &= s^-. \end{aligned} \tag{2.7}$$

Hence we get periodic solutions, by Colonius and Kliemann [13, Chapter 3, Proposition 3.2.2], $\varphi_{\mathbb{S}}(\cdot, s^+, u^+)\,,\ \varphi_{\mathbb{S}}(\cdot, s^-, u^-) \in \operatorname{int}({}_{\mathbb{S}}D) \subset \mathbb{S}^{n-1}$.

Step 1. Let $s_0 \in \operatorname{int}({}_{\mathbb{S}}D)$ and $l := \{\alpha s_0 \in \mathbb{R}^n \mid \alpha > 0\}$. Then for every $x_0 \in l$ we have $l \subset \overline{\mathcal{O}^+(x_0)}$.

For the proof of this claim, consider arbitrary points $x_0 = \alpha_0 s_0$, $x_1 = \alpha_1 s_0 \in l$. The strategy is to steer the system from s_0 to s^+, then to go k times through the periodic trajectory for u^+, then to steer the system to s^-, go ℓ times through the periodic trajectory for u^-, and finally steer back the system to s_0. The numbers $k, \ell \in \mathbb{N}$ will be adjusted such that the corresponding trajectories in $\mathbb{R}^n$ starting in x_0 approach x_1.

By local accessibility in $\operatorname{int}({}_{\mathbb{S}}D)$ there are times $\tau_1, \tau_2, \tau_3 > 0$ and controls $v^1, v^2, v^3 \in \mathcal{U}$ with

$$\begin{aligned} \varphi_{\mathbb{S}}\left(\tau_1, s_0, v^1\right) &= s^+, \\ \varphi_{\mathbb{S}}\left(\tau_2, s^+, v^2\right) &= s^-, \\ \varphi_{\mathbb{S}}\left(\tau_3, s^-, v^3\right) &= s_0. \end{aligned}$$

One finds for the system in $\mathbb{R}^n$ numbers $\beta_1, \beta_2, \beta_3 > 0$ with

$$\varphi(\tau_1, x_0, v^1) = \varphi(\tau_1, \alpha_0 s_0, v^1) = \beta_1 s^+,\ \varphi(\tau_2, s^+, v^2) = \beta_2 s^-,\ \varphi(\tau_3, s^-, v^3) = \beta_3 s_0.$$

Now define for $k, \ell \in \mathbb{N}$ a control function $w^{k,\ell}$ by

$$
\begin{aligned}
&w^{k,\ell}(t) = v^1(t) \text{ for } t \in [0, \tau_1],\\
&w^{k,\ell}(t) = u^+(t - (\tau_1 + (i-1)\sigma^+)) \text{ for } t \in (\tau_1 + (i-1)\sigma^+, \tau_1 + i\sigma^+],\ i = 1, \dots, k,\\
&w^{k,\ell}(t) = v^2(t - (\tau_1 + k\sigma^+)) \text{ for } t \in (\tau_1 + k\sigma^+, \tau_1 + k\sigma^+ + \tau_2],\\
&w^{k,\ell}(t) = u^-(t - (\tau_1 + k\sigma^+ + \tau_2 + (i-1)\sigma^-))\\
&\quad \text{for } t \in (\tau_1 + k\sigma^+ + \tau_2 + (i-1)\sigma^-, \tau_1 + k\sigma^+ + \tau_2 + i\sigma^-], \quad i = 1, \dots, \ell,\\
&w^{k,\ell}(t) = v^3(t - (\tau_1 + k\sigma^+ + \tau_2 + \ell\sigma^-)),\\
&\quad \text{for } t \in (\tau_1 + k\sigma^+ + \tau_2 + \ell\sigma^-, \tau_1 + k\sigma^+ + \tau_2 + \ell\sigma^- + \tau_3].
\end{aligned}
$$

The corresponding trajectory on $\mathbb{S}^{n-1}$ is periodic and satisfies

$$
\begin{aligned}
\varphi_{\mathbb{S}}\left(\tau_1 + i\sigma^+, s_0, w^{k,\ell}\right) &= s^+ \text{ for } i = 0, 1, \dots, k,\\
\varphi_{\mathbb{S}}\left(\tau_1 + k\sigma^+ + \tau_2 + i\sigma^-, s_0, w^{k,\ell}\right) &= s^- \text{ for } i = 0, 1, \dots, \ell,\\
\varphi_{\mathbb{S}}\left(\tau_1 + k\sigma^+ + \tau_2 + \ell\sigma^- + \tau_3, s_0, w^{k,\ell}\right) &= s_0.
\end{aligned}
$$

For $\varphi_{\mathbb{S}}\left(\tau_1 + i\sigma^+, s_0, w^{k,\ell}\right)$ one has

$$
\begin{aligned}
\varphi_{\mathbb{S}}\left(\tau_1 + i\sigma^+, s_0, w^{k,\ell}\right) &= \varphi_{\mathbb{S}}\left(i\sigma^+, \varphi_{\mathbb{S}}\left(\tau_1, s_0, v^1\right), u^+(t - (\tau_1 + (i-1)\sigma^+))\right)\\
&= \varphi_{\mathbb{S}}\left(i\sigma^+, s^+, u^+(t - (\tau_1 + (i-1)\sigma^+))\right)\\
&= \varphi_{\mathbb{S}}\left((i-1)\sigma^+, s(\sigma^+, s^+, u^+(t - (\tau_1 + (i-1)\sigma^+))\right),\\
&\qquad u^+(t - (\tau_1 + (i-2)\sigma^+))\Big)\\
&= \varphi_{\mathbb{S}}\left((i-1)\sigma^+, s^+, u^+(t - (\tau_1 + (i-2)\sigma^+))\right)\\
&= \vdots\\
&= s^+,
\end{aligned}
$$

in which to conclude this we use assumption (2.7) and the cocycle property (1.2). The other statements follow using the same properties.

For the corresponding trajectory on $\mathbb{R}^n$ one finds, using (1.7)

$$
\begin{aligned}
\varphi(\tau_1 + i\sigma^+, x_0, w^{k,\ell}) &= \left(\alpha^+\right)^i \beta_1 s^+ \text{ for } i = 0, 1, \dots, k,\\
\varphi(\tau_1 + k\sigma^+ + \tau_2 + i\sigma^-, x_0, w^{k,\ell}) &= \left(\alpha^-\right)^i \beta_2 \left(\alpha^+\right)^k \beta_1 s^- \text{ for } i = 0, 1, \dots, \ell,\\
\varphi(\tau_1 + k\sigma^+ + \tau_2 + \ell\sigma^- + \tau_3, x_0, w^{k,\ell}) &= \beta_3 \left(\alpha^-\right)^\ell \beta_2 \left(\alpha^+\right)^k \beta_1 s_0.
\end{aligned}
$$

Recall that our goal is to reach $x_1 = \alpha_1 s_0$ approximately. Fixed α^- we can consider the map $(\alpha_0, \alpha_0 + \delta_0) \longrightarrow \mathbb{R}$ given by $\alpha \longmapsto \frac{\log(\alpha)}{\log((\alpha^-)^{-1})}$, this map is continuous and increasing, so its range is an interval, thus we can choose $\alpha^+ \in (\alpha_0^+, \alpha_0^+ + \delta_0)$ such that $\frac{\log(\alpha^+)}{\log((\alpha^-)^{-1})}$ is irrational, so $\frac{-\log \alpha^-}{\log \alpha^+}$ is irrational.

We apply Lemma 2.2.3 with $a = \alpha^+$, $b = (\alpha^-)^{-1}$ and $c = \alpha_1 (\beta_3\beta_2\beta_1)^{-1}$. Thus for every $\varepsilon > 0$ there are $k, \ell \in \mathbb{N}$ with

$$\left| (\alpha^+)^k (\alpha^-)^\ell - \alpha_1 (\beta_3\beta_2\beta_1)^{-1} \right| < \varepsilon,$$

hence for all $\varepsilon > 0$ there are $k, \ell \in \mathbb{N}$ with

$$\left| \beta_3\beta_2\beta_1 (\alpha^+)^k (\alpha^-)^\ell - \alpha_1 \right| < \varepsilon.$$

With that we have

$$\begin{aligned} & \|\varphi(\tau_1 + k\sigma^+ + \tau_2 + \ell\sigma^- + \tau_3, x_0, w^{k,\ell}) - \alpha_1 s_0\| \\ = \; & \|\beta_3\beta_2\beta_1 (\alpha^+)^k (\alpha^-)^\ell s_0 - \alpha_1 s_0\| \\ = \; & |\beta_3\beta_2\beta_1 (\alpha^+)^k (\alpha^-)^\ell - \alpha_1| \|s_0\| \\ < \; & \varepsilon \|s_0\|. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\|\varphi(\tau_1 + k\sigma^+ + \tau_2 + \ell\sigma^- + \tau_3, x_0, w^{k,\ell}) - \alpha_1 s_0\|$ converges to zero, that is, the sequence $\left(\varphi(\tau_1 + k\sigma^+ + \tau_2 + \ell\sigma^- + \tau_3, x_0, w^{k,\ell})\right)$ converges to $\alpha_1 s_0$. Therefore, $x_1 = \alpha_1 s_0 \in \overline{\mathcal{O}^+(x_0)}$.

Step 2. If $x_1, x_2 \in \{\alpha s \in \mathbb{R}^n \mid \alpha > 0, s \in {}_{\mathbb{S}}D\}$, then $x_2 \in \overline{\mathcal{O}^+(x_1)}$.

Recall ${}_{\mathbb{S}}\mathcal{O}^+(s)$ is the positive orbit of s in the system (2.4).

Let $x_1, x_2 \in \{\alpha s \in \mathbb{R}^n \mid \alpha > 0, s \in {}_{\mathbb{S}}D\}$, hence there are $\alpha_1, \alpha_2 > 0$ and $s_1, s_2 \in {}_{\mathbb{S}}D$ with $x_1 = \alpha_1 s_1$ and $x_2 = \alpha_2 s_2$.

As $s_0 \in \operatorname{int}({}_{\mathbb{S}}D)$ by assumption (i) and Lemma 1.1.8 (i) there are a control u_1 and a time $t_1 \geq 0$ with $\varphi_{\mathbb{S}}(t_1, s_1, u_1) = s_0$, hence $\varphi(t_1, x_1, u_1) = \gamma_1 s_0 \in l$ for some $\gamma_1 > 0$. Since $s_2 \in \overline{{}_{\mathbb{S}}\mathcal{O}^+(s_0)}$ one finds, for $\varepsilon > 0$, a control u_ε and a time $t_\varepsilon \geq 0$ such that, for $s_\varepsilon := s(t_\varepsilon, s_0, u_\varepsilon)$,

$$\|s_\varepsilon - s_2\| < \varepsilon/\alpha_2 \text{ and } \|\alpha_2 s_\varepsilon - x_2\| = \|\alpha_2 s_\varepsilon - \alpha_2 s_2\| < \varepsilon.$$

The trajectory in $\mathbb{R}^n$ satisfies $\varphi(t_\varepsilon, s_0, u_\varepsilon) = \gamma_\varepsilon s_\varepsilon$ for some $\gamma_\varepsilon > 0$. By (1.7) it follows that

$$\varphi\left(t_\varepsilon, \frac{\alpha_2}{\gamma_\varepsilon} s_0, u_\varepsilon\right) = \frac{\alpha_2}{\gamma_\varepsilon} \gamma_\varepsilon s_\varepsilon = \alpha_2 s_\varepsilon,$$

that is

$$\left\| \varphi\left(t_\varepsilon, \frac{\alpha_2}{\gamma_\varepsilon} s_0, u_\varepsilon\right) - \alpha_2 s_2 \right\| < \varepsilon,$$

thus the sequence $\left(\varphi\left(t_\varepsilon, \frac{\alpha_2}{\gamma_\varepsilon} s_0, u_\varepsilon\right)\right)$ converges to $\alpha_2 s_2 = x_2$ when $\varepsilon \to 0$. Step 1 implies that $l \subset \overline{\mathcal{O}^+(\gamma_1 s_0)}$, thus $\alpha_2 s_2 \in \bar{l} \subset \overline{\mathcal{O}^+(\gamma_1 s_0)}$, as $\gamma_1 s_0 \in \mathcal{O}^+(x_1)$ follow that $x_2 = \alpha_2 s_2 \in \overline{\mathcal{O}^+(x_1)}$.

Step 3. We have shown that the cone $D' := \{\alpha s \in \mathbb{R}^n \mid \alpha > 0, s \in {}_{\mathbb{S}}D\}$ is a set of complete approximate controllability (property (ii) of the Definition 1.1.7).

Suppose there are $D"$ a set of approximate controllability in $\mathbb{R}^n$ such that $D' \subset D"$ so ${}_{\mathbb{S}}D = \pi_{\mathbb{S}}(D') \subset \pi_{\mathbb{S}}(D")$, but any set of approximate controllability in $\mathbb{R}^n$ projects to a set of approximate controllability in $\mathbb{S}^{n-1}$, and ${}_{\mathbb{S}}D$ is a maximal set of approximate controllability, then $\pi_{\mathbb{S}}(D") = {}_{\mathbb{S}}D$, so $D" \subset \pi_{\mathbb{S}}^{-1}({}_{\mathbb{S}}D) = D'$.

Finally, for every point $x \in D'$ there is a control u with $\varphi(t, x, u) \in D'$ for all $t \geq 0$, since this holds in ${}_{\mathbb{S}}D$. Hence the cone D' is a control set and it has a nonvoid interior. □

Note that in the Proposition 2.2.1 the projection of each control set D of the homogeneous bilinear control system (2.3) is contained in some control set ${}_{\mathbb{S}}D$ for the projected system in $\mathbb{S}^{n-1}$, so D is contained in the cone $\{\alpha s \in \mathbb{R}^n \mid \alpha > 0, s \in {}_{\mathbb{S}}D\}$, but as we will see ahead, the cone may not be a control set.

Step 1 in the proof above is based on the following lemma which uses a Diophantine approximation property, also known as Kronecker's theorem.

Lemma 2.2.3. *Let a, b, c be real numbers with $a, b > 1, c > 0$, and $\frac{\log b}{\log a} \in \mathbb{R} \setminus \mathbb{Q}$. Then for every $\delta > 0$ there are $k, \ell \in \mathbb{N}$ such that $\left|a^k b^{-\ell} - c\right| < \delta$.*

Proof. Since the logarithm is continuously invertible, it enough to show that for every $\varepsilon > 0$ there are $k, \ell \in \mathbb{N}$ with

$$\varepsilon > \left|\log(a^k b^{-\ell}) - \log c\right|.$$

In fact, it is enough to take $\varepsilon = \log\left(\frac{\delta}{c} + 1\right)$ to obtain the inequality of the lemma.

Then we need to show the above inequality. Note that

$$\varepsilon > \left|\log(a^k b^{-\ell}) - \log c\right| = \left|k \log a - \ell \log b - \log c\right|,$$

or, dividing by $\log a > 0$,

$$\left|k - \ell \frac{\log b}{\log a} - \frac{\log c}{\log a}\right| < \frac{\varepsilon}{\log a}.$$

We use the following Diophantine approximation result which is due to Tchebychef [40, Théorème, p. 679]: For any irrational number α and any $\beta \in \mathbb{R}$ the inequality

$$x\left|y - \alpha x - \beta\right| < 2$$

has an infinite number of solutions in $x \in \mathbb{N}, y \in \mathbb{Z}$. Observe that here also $y \in \mathbb{N}$ if $\alpha > 0$, since then $\operatorname{sgn}(y) = \operatorname{sgn}(\alpha x) = \operatorname{sgn}(x) = 1$. For an application to the problem above, let

$\alpha = \frac{\log b}{\log a} > 0,\ \beta = \frac{\log c}{\log a}, x = \ell, y = k$. One obtains that

$$\ell \left| k - \ell \frac{\log b}{\log a} - \frac{\log c}{\log a} \right| < 2$$

has an infinite number of solutions $k, \ell \in \mathbb{N}$. Choosing ℓ large enough such that $\frac{2 \log a}{\ell} < \varepsilon$ and dividing by ℓ one gets, as desired,

$$\left| k - \ell \frac{\log b}{\log a} - \frac{\log c}{\log a} \right| < \frac{2}{\ell} < \frac{\varepsilon}{\log a}.$$

$\square$

The Diophantine approximation result used above is closely related to a theorem due to Minkowski on inhomogeneous linear Diophantine approximation; cf. Cassels, [10, Theorem I in Chapter III]. Here the existence of integers x, y solving $x\,|y - \alpha x - \beta| < \frac{1}{4}$ is established, but not the existence of infinitely many pairs x, y with this property, as required for the proof above.

Remark 2.2.4. *Suppose that for a control set* $_{\mathbb{S}}D$ *on the unit sphere, every point in the interior is locally accessible and there are control values* $u^{\pm} \in \operatorname{int}(\Omega)$ *such that* $A(u^+)$ *has an eigenvalue* $\lambda^+ > 0$ *and* $A(u^-)$ *has an eigenvalue* $\lambda^- < 0$ *with eigenspaces satisfying* $E(\lambda^{\pm}) \cap \operatorname{int}(_{\mathbb{S}}D) \neq \varnothing$. *Then assumption (ii) of Theorem 2.2.2 holds. In fact, all points* $s^{\pm} \in E(\lambda^{\pm}) \cap \operatorname{int}(_{\mathbb{S}}D)$ *are equilibria for the induced system on* $\mathbb{S}^{n-1}$ *with* $A(u^{\pm})s^{\pm} = \lambda^{\pm} s^{\pm}$. *This implies for all* $\sigma^{\pm} > 0$ *and the constant controls* $u^{\pm} \in \Omega$ *that*

$$\begin{aligned} \varphi(\sigma^+, s^+, u^+) &= \alpha_0^+ s^+ \text{ with } \alpha_0^+ := e^{\lambda^+ \sigma^+} > 1, \\ \varphi(\sigma^-, s^-, u^-) &= \alpha^- s^- \text{ with } \alpha^- := e^{\lambda^- \sigma^-} < 1. \end{aligned}$$

This follows, since the solutions of $\dot{x} = A(u^{\pm})x,\ x(0) = s^{\pm}$, *are given by*

$$\varphi(t, s^{\pm}, u^{\pm}) = e^{A(u^{\pm})t} s^{\pm} = e^{\lambda^{\pm} t} s^{\pm}.$$

Varying σ^+, *we get that* $\varphi(\sigma^+, s^+, u^+) = \alpha^+ s^+$ *for all* $\alpha^+ \in (\alpha_0^+, \alpha_0^+ + \delta_0)$ *and some* $\delta_0 > 0$.

The following two examples illustrate Theorem 2.2.2. We consider problems in $\mathbb{R}^2$ where the induced system on the unit circle is not locally accessible. First let A be given in Jordan normal form $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and let the matrices B_1 and B_2 be diagonal. The situation is a bit more complicated than in Remark 2.2.4, since the intersections of the relevant eigenspaces with the unit sphere yield boundary points of the control set $_{\mathbb{S}}D$.

Example 2.2.5. *Consider a system of the form*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + u(t) \begin{bmatrix} b_{11} & 0 \\ 0 & b_{21} \end{bmatrix} + v(t) \begin{bmatrix} b_{12} & 0 \\ 0 & b_{22} \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix},$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ and control values $(u(t), v(t)) \in \Omega \subset \mathbb{R}^2$. This can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda_1 + b_{11}u + b_{12}v & 0 \\ 0 & \lambda_2 + b_{21}u + b_{22}v \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A(u,v) \begin{bmatrix} x \\ y \end{bmatrix}.$$

For all $(u,v) \in \Omega$ the eigenvalues $\mu_1(u,v) = \lambda_1 + b_{11}u + b_{12}v$ and $\mu_2(u,v) = \lambda_2 + b_{21}u + b_{22}v$ of $A(u,v)$ have the eigenspaces $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$, respectively. We are going to impose some conditions, under which the Theorem 2.2.2 is valid. Assume that there are control values satisfying

(a) $(u_1, v_1), (u_2, v_2) \in \Omega$ *with*

$$\mu_1(u_1, v_1) > 0,\ \mu_2(u_1, v_1) < 0 \text{ and } \mu_1(u_2, v_2) < 0,\ \mu_2(u_2, v_2) > 0. \tag{2.8}$$

(b) $(u_3, v_3) \in \Omega$ *with*

$$\mu_1(u_3, v_3) = 0 \text{ and } \mu_2(u_3, v_3) > 0. \tag{2.9}$$

(c) $(u_4, v_4) \in \Omega$ *with*

$$\mu_1(u_4, v_4) = 0 \text{ and } \mu_2(u_4, v_4) < 0. \tag{2.10}$$

For (u_1, v_1) the eigenspace $\mathbb{R} \times \{0\}$ is attracting and for (u_2, v_2) the eigenspace $\{0\} \times \mathbb{R}$ is attracting. One easily verifies that on the unit circle $\mathbb{S}^1$ there are four open and invariant control sets ${}_{\mathbb{S}}D_i, i = 1, \ldots, 4$, with nonvoid interior on the unit sphere separated by the four points in the intersection of the eigenspaces $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ with $\mathbb{S}^1$. The four points in this intersection are invariant for all (u,v), hence they are not locally accessible, while every point in the control sets is locally accessible.

Let $\tau_1 > 0$, and define $\tau_2 := \tau_1 \frac{\mu_1(u_1,v_1) - \mu_2(u_1,v_1)}{\mu_2(u_3,v_3)} > 0$ and

$$(u^+(t), v^+(t)) := \begin{cases} (u_1, v_1) & \textit{for} \quad t \in [0, \tau_1] \\ (u_3, v_3) & \textit{for} \quad t \in (\tau_1, \tau_2 + \tau_1] \end{cases}.$$

Fix a point $s^+ \in {}_{\mathbb{S}}D_i$. *Then it follows that*

$$\begin{aligned}\varphi(\tau_2+\tau_1, s^+, u^+, v^+) &= \varphi(\tau_2, \varphi(\tau_1, s^+, u_1, v_1), u_3, v_3) \\ &= \begin{bmatrix} e^0 & 0 \\ 0 & e^{\tau_2\mu_2(u_3,v_3)} \end{bmatrix} \begin{bmatrix} e^{\tau_1\mu_1(u_1,v_1)} & 0 \\ 0 & e^{\tau_1\mu_2(u_1,v_1)} \end{bmatrix} s^+ \\ &= e^{\tau_1\mu_1(u_1,v_1)} s^+.\end{aligned}$$

Since $\tau_1 > 0$ *is arbitrary, the first equality in* (2.6) *holds with* $\sigma^+ = \tau_2 + \tau_1$ *and* $\alpha^+ = e^{\tau_1\mu_1(u_1,v_1)} > 1$.

Analogously, fix a point $s^- \in {}_{\mathbb{S}}D_i$. *Define, with* $\tau_1 > 0$ *and* $\tau_3 := \tau_1 \frac{\mu_1(u_2,v_2)-\mu_2(u_2,v_2)}{\mu_2(u_4,v_4)} > 0$ *the control function*

$$(u^-(t), v^-(t)) = \begin{cases} (u_2, v_2) & \text{for} \quad t \in [0, \tau_1] \\ (u_4, v_4) & \text{for} \quad t \in (\tau_1, \tau_3 + \tau_1] \end{cases}.$$

Then it follows that

$$\begin{aligned}\varphi(\tau_3+\tau_1, s^-, u^-, v^-) &= \varphi(\tau_3, \varphi(\tau_1, s^-, u_2, v_2), u_4, v_4) \\ &= \begin{bmatrix} e^0 & 0 \\ 0 & e^{\tau_3\mu_2(u_4,v_4)} \end{bmatrix} \begin{bmatrix} e^{\tau_1\mu_1(u_2,v_2)} & 0 \\ 0 & e^{\tau_1\mu_2(u_2,v_2)} \end{bmatrix} s^- \\ &= e^{\tau_1\mu_1(u_2,v_2)} s^-.\end{aligned}$$

Thus also the second equality in (2.6) *holds with* $\sigma^- = \tau_3 + \tau_1$ *and* $\alpha^- = e^{\tau_1\mu_1(u_2,v_2)} < 1$. *Now Theorem* 2.2.2 *implies that there are four control set in* $\mathbb{R}^2$ *given by the interiors of the four quadrants.*

Observe that conditions (2.8), (2.9), *and* (2.10) *are satisfied in the simple example with* $A(u,v) = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$ *and* $\Omega = [-1,1] \times [-1,1]$. *Then* $\mu_1(u,v) = u$, $\mu_2(u,v) = v$, *and one may choose*

$$(u_1, v_1) = (1, -1), \ (u_2, v_2) = (-1, 1), \ (u_3, v_3) = (0, 1) \text{ and } (u_4, v_4) = (0, -1).$$

The next example shows that the situation is quite different if A is a two-dimensional Jordan block; in particular, for scalar controls is suffice to verify assumption (2.6) in Theorem 2.2.2 for a control set ${}_{\mathbb{S}}D \neq \mathbb{S}^1$.

Example 2.2.6. *Consider*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} + u(t) \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{11} \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix},$$

with $\lambda \in \mathbb{R}$ *and* $u(t) \in \Omega$. *The system can be written as*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda + b_{11}u & 1 + b_{12}u \\ 0 & \lambda + b_{11}u \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A(u) \begin{bmatrix} x \\ y \end{bmatrix}.$$

For all $u \in \Omega$ *the eigenvalue* $\mu(u) = \lambda + b_{11}u$ *has the eigenspace* $\mathbb{R} \times \{0\}$. *The intersection of the unit circle with the eigenspace is given by* $\{(1,0)^\top, (-1,0)^\top\}$, *which are fixed under any control for the projected system. Suppose that* $b_{12} \neq 0$ *and* Ω *contains the two points* $u_1 := 0$ *and* $u_2 := -2/b_{12}$, *and write* $\mu_1 = \mu(u_1) = \lambda$ *and* $\mu_2 = \mu(u_2) = \lambda - 2\frac{b_{11}}{b_{12}}$. *Thus we consider the two differential equations*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} -\mu_2 & 1 \\ 0 & -\mu_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.11}$$

The solutions of (2.11) are given by

$$\psi_1(t, x_0, y_0) = e^{\mu_1 t} \begin{bmatrix} x_0 + ty_0 \\ y_0 \end{bmatrix} \text{ and } \psi_2(t, x_0, y_0) = e^{\mu_2 t} \begin{bmatrix} x_0 - ty_0 \\ y_0 \end{bmatrix},$$

respectively. For the projected systems on the unit circle the trajectory on the upper half-plane of the first equation tends for $t \to \infty$ *to* $(1,0)$ *and for* $t \to -\infty$ *to* $(-1,0)$. *The trajectory for the second equation moves in the opposite direction. This proves that the open upper semicircle on* $\mathbb{S}^1$ *is an invariant control set* ${}_{\mathbb{S}}D_1$. *Analogously, also the open lower semicircle on* $\mathbb{S}^1$ *is an invariant control set* ${}_{\mathbb{S}}D_2$.

In order to verify the conditions in (2.6) fix a point $s^+ \in {}_{\mathbb{S}}D_1$. *Let* $\tau > 0$ *and define*

$$u^+(t) = \begin{cases} u_1 & \text{for } t \in [0, \tau] \\ u_2 & \text{for } t \in (\tau, 2\tau] \end{cases}.$$

It follows that

$$\varphi(2\tau, s^+, u^+) = \psi_2(\tau, \psi_1(\tau, s^+)) = e^{\mu_2\tau + \mu_1\tau} s^+.$$

Then $\alpha^+ = e^{\mu_2\tau+\mu_1\tau} > 1$ *if and only if* $\mu_2 + \mu_1 = 2\lambda - 2\frac{b_{11}}{b_{12}} > 0$, *i.e.,* $\lambda > \frac{b_{11}}{b_{12}}$. *Similarly, we can find conditions for* $\alpha^- < 1$. *The control sets on the unit sphere do not change if we add a third control value* u_3 *which will be specified in a moment. Repeating the derivation above, we find with* $\mu_3 := \mu(u_3)$ *that* $\alpha^- := e^{\mu_3\tau+\mu_2\tau} < 1$ *if and only if* $\mu_3 + \mu_2 = \lambda + b_{11}u_3 + \lambda - 2\frac{b_{11}}{b_{12}} < 0$. *This is equivalent to*

$$u_3 b_{11} < 2\frac{b_{11}}{b_{12}} - 2\lambda. \tag{2.12}$$

We conclude that condition (2.6) holds if $\lambda > \frac{b_{11}}{b_{12}}$ *for* $\Omega = \{u_1, u_2, u_3\}$ *with* $u_1 = 0$ *and* $u_2 = -\frac{2}{b_{12}}$, *and* u_3 *satisfying (2.12). Then there are two invariant control sets with* *nonvoid interior in $\mathbb{R}^2$ given by the open upper and lower half-planes. Observe that these conditions hold, e.g., for*

$$\lambda = 1, b_{11} = 1, b_{12} = 2, \text{ and } u_1 = 0, u_2 = -1, u_3 < -1.$$

Next we impose stronger assumptions on the homogeneous bilinear control system (1.6). Suppose that the accessibility rank condition holds on all of $\mathbb{P}^{n-1}$,

$$\dim \mathcal{LA}\{H(\cdot, u); u \in \Omega\}(p) = n - 1 \text{ for all } p \in \mathbb{P}^{n-1}. \tag{2.13}$$

Then by Colonius and Kliemann [13, Theorem 7.1.1] there are k_0 control sets with nonvoid interior in $\mathbb{P}^{n-1}$ denoted by ${}_{\mathbb{P}}D_1, \ldots, {}_{\mathbb{P}}D_{k_0}, 1 \leq k_0 \leq n$. Exactly one of these control sets is an invariant control set.

Remark 2.2.7. *Braga Barros and San Martin [7] use the classification of semisimple Lie groups acting transitively on projective space $\mathbb{P}^{n-1}$ (cf. Boothby and Wilson [6]) to determine the number $k_0 \in \{1, \ldots, n\}$ of control sets ${}_{\mathbb{P}}D_i$ in projective space (it is either equal to n, $n/2$, or $n/4$).*

The proof of the next proposition uses arguments from Bacciotti and Vivalda [3, Proposition 2].

Proposition 2.2.8. *If accessibility rank condition (2.13) holds for the projected system on $\mathbb{P}^{n-1}$, it also holds for the projected system on $\mathbb{S}^{n-1}$.*

Proof. For the sake of simplicity we prove the rank condition for the North Pole of $\mathbb{S}^{n-1}$ given by $\bar{z}_0 = (0, \ldots, 0, 1)$. By assumption, the rank of the Lie algebra of the system on $\mathbb{P}^{n-1}$ is $n - 1$ on all points of $\mathbb{P}^{n-1}$. Consider the point $x_0 = [0 : \cdots : 0 : 1] \in \mathbb{P}^{n-1}$. Thus there exist $n - 1$ matrices $A_1, \ldots, A_{n-1}$ in the Lie algebra generated by the system on $\mathbb{R}^n \setminus \{0\}$ such that for the induced vector fields $A_1^\flat, \ldots, A_{n-1}^\flat$ in the Lie algebra for the system on $\mathbb{P}^{n-1}$ one obtains that the rank of the family $\left(A_1^\flat(x_0), \ldots, A_{n-1}^\flat(x_0)\right)$ is $n - 1$. Now [3, formula (5)] shows the following formula for the local expression of this family, which has the form $\left(A_1^n(z_0), \ldots, A_{n-1}^n(z_0)\right)$ with $z_0 = (0, \ldots, 0)$; let $a_1^k(\bar{z}_0), \ldots, a_n^k(\bar{z}_0)$ denote the n components of $A_k \bar{z}_0$. Then, for $k = 1, \ldots, n - 1$,

$$A_k^n(z_0) = (a_1^k(\bar{z}_0), \ldots, a_{n-1}^k(\bar{z}_0))^\top - a_n^k(\bar{z}_0) z_0 = (a_1^k(\bar{z}_0), \ldots, a_{n-1}^k(\bar{z}_0))^\top.$$

So $A_k^n(z_0)$ is the vector whose components are equal to the first $n - 1$ components of the last column of the matrix A_k.

On the other hand, the projections on $\mathbb{S}^{n-1}$ of the linear vector fields for the matrices $A_1, \ldots, A_{n-1}$ are the vector fields (cf.(2.4))

$$A_k^\circ(x) = A_k x - x^\top A x \cdot x, \quad x \in \mathbb{S}^{n-1}.$$

Thus we get, for $k = 1, \dots, n-1$

$$A_k^\circ(\bar{z}_0) = A_k\bar{z}_0 - \bar{z}_0^\top A_k\bar{z}_0 \cdot \bar{z}_0 = (a_1^k(\bar{z}_0), \dots, a_{n-1}^k(\bar{z}_0), a_n^k(z_0) - \bar{z}_0^\top A_k\bar{z}_0)^\top,$$

so the $n-1$ first components of $A_k^\circ(\bar{z}_0)$ are equal to the components of $A_k^n(z_0)$. This implies that the vectors $A_1^\circ(\bar{z}_0), \dots, A_{n-1}^\circ(\bar{z}_0)$ are linearly independent. □

Next we analyze the relations between the control sets for the projected system (2.5) on projective space $\mathbb{P}^{n-1}$ and the system (2.4) on the unit sphere $\mathbb{S}^{n-1}$.

In the next results we will consider the projection of sphere $\mathbb{S}^{n-1}$ on the projective space $\mathbb{P}^{n-1}$, but it will denoted by $\pi_\mathbb{P}$ as the projection of $\mathbb{R}^n$ on the projective space $\mathbb{P}^{n-1}$.

We will frequently use the following elementary facts that follow from (1.7).

Lemma 2.2.9. *(i) Let $s_1, s_2 \in \mathbb{S}^{n-1}$, if $s_2 \in {}_\mathbb{S}\mathcal{O}^+(s_1)$, then $-s_2 \in {}_\mathbb{S}\mathcal{O}^+(-s_1)$.*

(ii) If on $\mathbb{P}^{n-1}$ the point $\pi_\mathbb{P}(s_2) \in {}_\mathbb{P}\mathcal{O}^+(\pi_\mathbb{P}(s_1))$, then one of these is valid, $s_2 \in {}_\mathbb{S}\mathcal{O}^+(s_1)$ or $-s_2 \in {}_\mathbb{S}\mathcal{O}^+(s_1)$.

(iii) If $\pi_\mathbb{P}(s_2) \in \overline{{}_\mathbb{P}\mathcal{O}^+(\pi_\mathbb{P}(s_1))}$, then $s_2 \in \overline{{}_\mathbb{S}\mathcal{O}^+(s_1)}$ or $-s_2 \in \overline{{}_\mathbb{S}\mathcal{O}^+(s_1)}$.

Proof. (i) If $s_2 = \varphi_\mathbb{S}(t, s_1, u)$, so $-s_2 = -\varphi_\mathbb{S}(t, s_1, u) = \varphi_\mathbb{S}(t, -s_1, u)$, so $-s_1 \in \mathcal{O}_\mathbb{S}^+(-s_2)$.

(ii) If $\pi_\mathbb{P}(s_2) = \varphi_\mathbb{P}(t, \pi_\mathbb{P}(s_1), u) = \pi_\mathbb{P}(\varphi_\mathbb{S}(t, s_1, u))$ then

$$\varphi_\mathbb{S}(t, s_1, u) \in \pi_\mathbb{P}^{-1}(\pi_\mathbb{P}(s_2)) = \{-s_2,\ s_2\}.$$

Thus on $\mathbb{S}^{n-1}$ at least one of the points s_2 or $-s_2$ can be reached from s_1.

(iii) Let $(\varphi_\mathbb{P}(t_n, \pi_\mathbb{P}(s_2), u_n))_{n\in\mathbb{N}}$ be a sequence of points of ${}_\mathbb{S}\mathcal{O}^+(\pi_\mathbb{P}(s_2))$ that converges to s_1, when $n \to \infty$. By item (ii) $s_n =: \varphi_\mathbb{S}(t_n, s_2, u_n) \in \{-s_n, s_n\}$, thus define

$$A^+ = \{s_n \mid \varphi_\mathbb{S}(t_n, s_2, u_n) = s_n\},$$

$$A^- = \{s_n \mid \varphi_\mathbb{S}(t_n, s_2, u_n) = -s_n\},$$

in which of these sets has infinite elements. Suppose, without loss of generality, that A^+ has infinite elements, then we can consider a subsequence $(\varphi_\mathbb{S}(t_{n_k}, s_2, u_{n_k}))$ of elements A^+ which converges to s_1. In fact, if V is a neighborhood of s_1, so $\pi_\mathbb{P}(V)$ is a neighborhood of $\pi_\mathbb{P}(s_1)$, then there are $n_0 \in \mathbb{N}$ such that $\varphi_\mathbb{P}(t_n, \pi_\mathbb{P}(s_2), u_n) \in \pi_\mathbb{P}(V)$ for all $n \geq n_0$, thus for all $n_k \geq n_0$ one has $\varphi_\mathbb{S}(t_{n_k}, s_2, u_{n_k}) \in V$. As $A^+ \subset {}_\mathbb{S}\mathcal{O}^+(s_2)$ it follows that $s_1 \in \overline{{}_\mathbb{S}\mathcal{O}^+(s_2)}$. □

The proof of the following lemma is modeled after Bacciotti and Vivalda [3, Lemma 3], where controllable systems are analyzed.

Lemma 2.2.10. *(i) Let ${}_\mathbb{S}D$ be a control set on $\mathbb{S}^{n-1}$. Then the projection of ${}_\mathbb{S}D$ to $\mathbb{P}^{n-1}$ is contained in a control set ${}_\mathbb{P}D$.*

(ii) Assume that the accessibility rank condition (2.13) on $\mathbb{P}^{n-1}$ holds and consider a control set ${}_{\mathbb{P}}D_i$ on $\mathbb{P}^{n-1}$. Suppose that there is $s_0 \in \mathbb{S}^{n-1}$ such that $\pi_{\mathbb{P}}(s_0) \in \operatorname{int}({}_{\mathbb{P}}D_i)$ and $-s_0$ can be reached from s_0. Then there exists a control set ${}_{\mathbb{S}}D$ on $\mathbb{S}^{n-1}$ containing $A := \{s \in \mathbb{S}^{n-1} \mid \pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i\}$.

Proof. Assertion (i) follows similarly to Proposition 2.2.1 .

(ii) Recall that $\varphi_{\mathbb{P}}(t, \pi_{\mathbb{P}}(s), u)$ denotes the solution of the projected system (2.5) on the projective space.

For each $s \in A$ its projection $\pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i$, as ${}_{\mathbb{P}}D_i$ is a control set for the projected system (2.5), there are a solution such that $\varphi_{\mathbb{P}}(t, \pi_{\mathbb{P}}(s), u) \in {}_{\mathbb{P}}D_i$ for all $t \geq 0$. Consider the solution $\varphi_{\mathbb{S}}(t, s, u)$, and noticed that $\pi_{\mathbb{P}}(\varphi_{\mathbb{S}}(t, s, u)) = \varphi_{\mathbb{P}}(t, \pi_{\mathbb{P}}(s), u)$ for all $t \geq 0$. Thus $\varphi_{\mathbb{S}}(t, s, u) \in A$ for all $t \geq 0$.

Now let $s_1, s_2 \in A$. We have to show that $s_2 \in \overline{{}_{\mathbb{S}}\mathcal{O}^+(s_1)}$.

From assumption (ii) $-s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(s_0)$ which implies $s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(-s_0)$. So by Property 1.1.6, ${}_{\mathbb{S}}\mathcal{O}^+(s_0) = {}_{\mathbb{S}}\mathcal{O}^+(-s_0)$.

Since $\pi_{\mathbb{P}}(s_1), \pi_{\mathbb{P}}(s_2) \in {}_{\mathbb{P}}D_i$ it follows that $\pi_{\mathbb{P}}(s_2) \in \overline{{}_{\mathbb{P}}\mathcal{O}^+(\pi_{\mathbb{P}}(s_1))}$, then $s_2 \in \overline{{}_{\mathbb{S}}\mathcal{O}^+(s_1)}$ or $-s_2 \in \overline{{}_{\mathbb{S}}\mathcal{O}^+(s_1)}$. In the first case we are done.

In the second case it follows that $s_2 \in \overline{{}_{\mathbb{S}}\mathcal{O}^+(-s_1)}$. As $\pi_{\mathbb{P}}(s_1) \in {}_{\mathbb{P}}D_i$ and $\pi_{\mathbb{P}}(s_0) \in \operatorname{int}({}_{\mathbb{P}}D_i)$, Lemma 1.1.8 implies that $\pi_{\mathbb{P}}(s_0) \in {}_{\mathbb{P}}\mathcal{O}^+(\pi_{\mathbb{P}}(s_1))$, hence $s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(s_1)$ or $-s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(s_1)$. By above observation $-s_0, s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(s_1)$.

Since $\pi_{\mathbb{P}}(s_0), \pi_{\mathbb{P}}(s_2) \in {}_{\mathbb{P}}D_i$ it follows that $s_2 \in \overline{{}_{\mathbb{S}}\mathcal{O}^+(s_0)}$ or $-s_2 \in \overline{{}_{\mathbb{S}}\mathcal{O}^+(s_0)}$. In any case we have

$$s_2 \in \overline{{}_{\mathbb{S}}\mathcal{O}^+(s_0)} \subset \overline{{}_{\mathbb{S}}\mathcal{O}^+(s_1)}.$$

□

We get the following result characterizing the relation between the control sets on projective space and the control sets on the unit sphere.

Theorem 2.2.11. *Suppose that accessibility rank condition (2.13) holds for the projected system (2.5) on projective space $\mathbb{P}^{n-1}$. Let ${}_{\mathbb{P}}D_i$ $i = 1, \ldots, k_0$ all the control sets with nonvoid interior of the projected system (2.5).*

(i) If there is $s_0 \in \mathbb{S}^{n-1}$ with $\pi_{\mathbb{P}}(s_0) \in \operatorname{int}({}_{\mathbb{P}}D_i)$ such that $-s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(s_0)$, then ${}_{\mathbb{S}}D := \{s \in \mathbb{S}^{n-1} \mid \pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i\}$ is the unique control set on $\mathbb{S}^{n-1}$ which projects to ${}_{\mathbb{P}}D_i$.

(ii) For every control set ${}_{\mathbb{P}}D_i, i \in \{1, \ldots, k_0\}$, there are at most two control sets ${}_{\mathbb{S}}D$ and ${}_{\mathbb{S}}D'$ on $\mathbb{S}^{n-1}$ with nonvoid interior such that

$$\{s \in \mathbb{S}^{n-1} \mid \pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i\} = {}_{\mathbb{S}}D \cup {}_{\mathbb{S}}D', \tag{2.14}$$

and ${}_{\mathbb{S}}D = -\,{}_{\mathbb{S}}D'$.

(iii) There are k_1 control sets with nonvoid interior on $\mathbb{S}^{n-1}$ denoted by ${}_{\mathbb{S}}D_1, \ldots, {}_{\mathbb{S}}D_{k_1}$ with $1 \leq k_1 \leq 2k_0 \leq 2n$. At most two of the sets ${}_{\mathbb{S}}D_i$ are invariant control sets.

Proof. (i) Suppose that there is $\pi_{\mathbb{P}}(s_0) \in \operatorname{int}({}_{\mathbb{P}}D_i)$ with $-s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(s_0)$. By Lemma 2.2.10 (ii) there is a control set ${}_{\mathbb{S}}D$ on the unit sphere containing $\{s \in \mathbb{S}^{n-1} \mid \pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i\}$, hence the projection of ${}_{\mathbb{S}}D$ to projective space contains the control set ${}_{\mathbb{P}}D_i$. Using Lemma 2.2.10 (i) one concludes that $\pi_{\mathbb{P}}({}_{\mathbb{S}}D)$ is contained in some control set of projected system (2.5) as $\pi_{\mathbb{P}}({}_{\mathbb{S}}D) \cap {}_{\mathbb{P}}D_i \neq \varnothing$. Thus $\pi_{\mathbb{P}}({}_{\mathbb{S}}D) \subset {}_{\mathbb{P}}D_i$, so ${}_{\mathbb{S}}D = \{s \in \mathbb{S}^{n-1} \mid \pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i\}$.

(ii) Fix a point $s_0 \in \mathbb{S}^{n-1}$ with $\pi_{\mathbb{P}}(s_0) \in \operatorname{int}({}_{\mathbb{P}}D_i)$ and define

$$A^+ := \left\{ s \in \mathbb{S}^{n-1} \,\middle|\, \pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i \text{ and } s \in {}_{\mathbb{S}}\mathcal{O}^+(s_0) \cap {}_{\mathbb{S}}\mathcal{O}^-(s_0) \right\},$$
$$A^- := \left\{ s \in \mathbb{S}^{n-1} \,\middle|\, \pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i \text{ and } -s \in {}_{\mathbb{S}}\mathcal{O}^+(s_0) \cap {}_{\mathbb{S}}\mathcal{O}^-(s_0) \right\}.$$

First we will prove that these sets are contained in some control set.

In fact, for each $s \in A^+$, $s \in \mathcal{O}^+_{\mathbb{S}}(s_0) \cap \mathcal{O}^-_{\mathbb{S}}(s_0)$, so there are control functions $u, v \in \mathcal{U}_{pc}$, and times $t_u, t_v \geq 0$ such that $\varphi_{\mathbb{S}}(t_u, s, u) = s_0$ and $\varphi_{\mathbb{S}}(t_v, s_0, v) = s$. Consider $(t_u + t_v)$-periodic control w given by

$$w(t) = \begin{cases} u(t), & \text{if } t \in [0, t_u) \\ v(t_u - t), & \text{if } t \in [t_u, t_u + t_v]. \end{cases}$$

Thus $\varphi_{\mathbb{S}}(t_u + t_v, s, w) = s$ and $\varphi_{\mathbb{S}}(t_u + t_v, s_0, w) = s_0$, so $\varphi_{\mathbb{S}}(t, s, w) \in A^+$ for all $t \geq 0$. Moreover, $s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(s)$ then ${}_{\mathbb{S}}\mathcal{O}^+(s_0) \subset {}_{\mathbb{S}}\mathcal{O}^+(s)$ and $A^+ \subset \mathcal{O}^+_{\mathbb{S}}(s_0) \cap \mathcal{O}^-_{\mathbb{S}}(s_0) \subset \mathcal{O}^+_{\mathbb{S}}(s_0) \subset \mathcal{O}^+_{\mathbb{S}}(s)$, this proves the claim. The same arguments prove that A^- is contained in some control set ${}_{\mathbb{S}}D'$.

Now we will prove that there are at most two control sets on $\mathbb{S}^{n-1}$ which projects to ${}_{\mathbb{P}}D_i$.

We have ${}_{\mathbb{P}}D_i = \overline{{}_{\mathbb{P}}\mathcal{O}^+(\pi_{\mathbb{P}}(s_0))} \cap {}_{\mathbb{P}}\mathcal{O}^-(\pi_{\mathbb{P}}(s_0))$. Moreover, by Lemma 1.1.8 every point $\pi_{\mathbb{P}}(s) \in \operatorname{int}({}_{\mathbb{P}}D_i)$ satisfies $\pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}\mathcal{O}^+(\pi_{\mathbb{P}}(s_0))$, so $\pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}\mathcal{O}^+(\pi_{\mathbb{P}}(s_0)) \cap {}_{\mathbb{P}}\mathcal{O}^-(\pi_{\mathbb{P}}(s_0))$.

Therefore, for each $s \in \mathbb{S}^{n-1}$ with $\pi_{\mathbb{P}}(s) \in \operatorname{int}({}_{\mathbb{P}}D_i)$ satisfies one of those statements:

$$\begin{array}{rclcrcl} s & \in & {}_{\mathbb{P}}\mathcal{O}^+(s_0) & \text{and} & -s & \in & {}_{\mathbb{S}}\mathcal{O}^-(s_0) \\ -s & \in & {}_{\mathbb{P}}\mathcal{O}^+(s_0) & \text{and} & s & \in & {}_{\mathbb{S}}\mathcal{O}^-(s_0) \\ s & \in & {}_{\mathbb{P}}\mathcal{O}^+(s_0) & \text{and} & s & \in & {}_{\mathbb{S}}\mathcal{O}^-(s_0) \\ -s & \in & {}_{\mathbb{P}}\mathcal{O}^+(s_0) & \text{and} & -s & \in & {}_{\mathbb{S}}\mathcal{O}^-(s_0). \end{array}$$

If there is $s \in {}_{\mathbb{S}}\mathcal{O}^+(s_0)$ with $-s \in {}_{\mathbb{S}}\mathcal{O}^-(s_0)$, then $s \in {}_{\mathbb{S}}\mathcal{O}^-(-s_0)$, which implies $-s_0 \in {}_{\mathbb{S}}\mathcal{O}^+(s_0)$. The assertion follows by item (i). The same arguments can be applied if there is s with $-s \in {}_{\mathbb{S}}\mathcal{O}^+(s_0)$ and $s \in {}_{\mathbb{S}}\mathcal{O}^-(s_0)$. Hence we may assume that either $s \in {}_{\mathbb{S}}\mathcal{O}^+(s_0) \cap$

${}_{\mathbb{S}}\mathcal{O}^-(s_0)$ or $-s \in {}_{\mathbb{S}}\mathcal{O}^+(s_0) \cap {}_{\mathbb{S}}\mathcal{O}^-(s_0)$. This shows that

$$\{s \in \mathbb{S}^{n-1} \mid \pi_{\mathbb{P}}(s) \in \mathrm{int}\,({}_{\mathbb{P}}D_i)\} \subset A^+ \cup A^- \subset ({}_{\mathbb{S}}D) \cup ({}_{\mathbb{S}}D') \,.$$

It follows that $\{s \in \mathbb{S}^{n-1} \mid \pi_{\mathbb{P}}(s) \in {}_{\mathbb{P}}D_i\} \subset \overline{{}_{\mathbb{S}}D} \cup \overline{{}_{\mathbb{S}}D'}$, since $\pi_{\mathbb{P}}$ is an open map and ${}_{\mathbb{P}}D_i \subset \overline{\mathrm{int}\,({}_{\mathbb{P}}D_i)}$. By Lemma 2.2.10 (i) the projections of ${}_{\mathbb{S}}D$ and ${}_{\mathbb{S}}D'$ to $\mathbb{P}^{n-1}$ are contained in ${}_{\mathbb{P}}D_i$, hence (2.14) follows. The same arguments with $-s_0$ instead of s_0 implies that ${}_{\mathbb{S}}D = -{}_{\mathbb{S}}D'$. If ${}_{\mathbb{S}}D$ or ${}_{\mathbb{S}}D'$ is an invariant control set, then also ${}_{\mathbb{P}}D_i$ is an invariant control set, hence there are at most two invariant control sets on $\mathbb{S}^{n-1}$.

(iii) This is a consequence of assertion (ii). □

The following definitions introduce several spectral concepts. The Lyapunov exponent for $(u, x) \in \mathcal{U} \times (\mathbb{R}^n \setminus \{0\})$ is

$$\lambda(u, x) = \limsup_{\tau\to\infty} \frac{1}{\tau} \log \|\varphi(\tau, x, u)\| \,.$$

Definition 2.2.12. *For controls set in $\mathbb{S}^{n-1}$ and $\mathbb{P}^{n-1}$ the* ***Floquet spectrum*** *and* ***Lyapunov spectrum*** *are given as:*

(i) The Floquet spectrum of a control set ${}_{\mathbb{S}}D$ on the unit sphere $\mathbb{S}^{n-1}$ is

$$\begin{aligned}\Sigma_{Fl}({}_{\mathbb{S}}D) \;=\; & \{\lambda(u, x) \mid \pi_{\mathbb{S}}(x) \in \mathrm{int}({}_{\mathbb{S}}D),\; u \in \mathcal{U}_{pc}\ \tau\text{-periodic with}\\ & \varphi_{\mathbb{S}}(\tau, \pi_{\mathbb{S}}(x), u) = \pi_{\mathbb{S}}(x)\}\,.\end{aligned}$$

(ii) The Floquet spectrum of a control set ${}_{\mathbb{P}}D$ on projective space $\mathbb{P}^{n-1}$ is

$$\begin{aligned}\Sigma_{Fl}({}_{\mathbb{P}}D) \;=\; & \{\lambda(u, x) \mid \pi_{\mathbb{P}}(x) \in \mathrm{int}({}_{\mathbb{P}}D),\; u \in \mathcal{U}_{pc}\ \tau\text{-periodic with}\\ & \varphi_{\mathbb{P}}(\tau, \pi_{\mathbb{P}}(x), u) = \pi_{\mathbb{P}}(x)\}\,.\end{aligned}$$

(iii) The Lyapunov spectrum of a control set ${}_{\mathbb{P}}D$ on projective space $\mathbb{P}^{n-1}$ is

$$\Sigma_{Ly}({}_{\mathbb{P}}D) = \left\{\lambda(u, x) \mid \varphi_{\mathbb{P}}(t, \pi_{\mathbb{P}}(x), u) \in \overline{{}_{\mathbb{P}}D}\ \text{ for all }\ t \geq 0\right\}.$$

In the τ-periodic case considered here the Lyapunov exponents satisfy $\lambda(u, x) = \frac{1}{\tau}\log\|\varphi(\tau, x, u)\|$ for $\|x\| = 1$ and coincide with the Floquet exponents (cf. Teschl [41, §3.6]).

Remark 2.2.13. *Suppose that the accessibility rank condition in $\mathbb{P}^{n-1}$ holds and denote the system semigroup of the system on $\mathbb{P}^{n-1}$ by $\mathcal{S}^{\mathrm{hom}}$. Colonius and Kliemann [13, Corollary 7.3.18] implies that the Floquet spectrum of a control set ${}_{\mathbb{P}}D$ consists of the numbers $\frac{1}{\tau}\log|\rho|$ where ρ is a real eigenvalue of an element $\Phi_u(\tau, 0)$ with eigenspace $\mathbf{E}(\Phi_u(\tau, 0); \rho) \subset \mathrm{int}({}_{\mathbb{P}}D)$ and corresponding element $h(u) \in \mathcal{S}^{\mathrm{hom}}_{\tau} \cap \mathrm{int}(\mathcal{S}^{\mathrm{hom}}_{\leq\tau+1})$.*

Proposition 2.2.14. *If ${}_{\mathbb{S}}D$ is a control set with nonvoid interior on $\mathbb{S}^{n-1}$ that projects to a control set ${}_{\mathbb{P}}D$ in $\mathbb{P}^{n-1}$, then*

$$\Sigma_{Fl}({}_{\mathbb{S}}D) = \Sigma_{Fl}({}_{\mathbb{P}}D).$$

Proof. Suppose that $\lambda \in \Sigma_{Fl}({}_{\mathbb{S}}D)$, so $\lambda = \lambda(u,x)$, thus $\pi_{\mathbb{S}}(x) \in \mathrm{int}({}_{\mathbb{S}}D)$, u is $\tau-$periodic and $\pi_{\mathbb{S}}(x) = \varphi_{\mathbb{S}}(\tau, \pi_{\mathbb{S}}(x), u)$. One has $\pi_{\mathbb{P}}(\pi_{\mathbb{S}}(x)) \in \mathrm{int}({}_{\mathbb{P}}D)$, but $\pi_{\mathbb{P}}(\pi_{\mathbb{S}}(x)) = \pi_{\mathbb{P}}(x)$ and $\pi_{\mathbb{P}}\left(\varphi_{\mathbb{S}}(\tau, \pi_{\mathbb{S}}(x), u)\right) = \varphi_{\mathbb{P}}(\tau, \pi_{\mathbb{P}}(x), u)$, then $\lambda \in \Sigma_{Fl}({}_{\mathbb{P}}D)$.

Now, suppose that $\lambda \in \Sigma_{Fl}({}_{\mathbb{P}}D)$, then $\lambda = \lambda(u,x)$ and u is τ-periodic control such that $\pi_{\mathbb{P}}(x) \in \mathrm{int}\,({}_{\mathbb{P}}D)$ and $\varphi_{\mathbb{P}}(\tau, \pi_{\mathbb{P}}(x), u) = \pi_{\mathbb{P}}(x)$. We may suppose that $x \in \mathbb{S}^{n-1}$. By Theorem 2.2.11 $x \in {}_{\mathbb{S}}D$ or $-x \in {}_{\mathbb{S}}D$. Moreover, $\pi_{\mathbb{P}}^{-1}\{\mathrm{int}({}_{\mathbb{P}}D)\}$ is a open subset of ${}_{\mathbb{S}}D \cup -{}_{\mathbb{S}}D$, so $x \in \mathrm{int}({}_{\mathbb{S}}D)$ or $-x \in \mathrm{int}({}_{\mathbb{S}}D)$. In the first case, if $\varphi(\tau, x, u) = \alpha x$ with $\alpha > 0$, so $\pi_{\mathbb{S}}(\varphi(\tau, x, u)) = \varphi_{\mathbb{S}}(\tau, x, u) = x$, then $\lambda(u,x) = \frac{1}{\tau}\log\alpha \in \Sigma_{Fl}({}_{\mathbb{S}}D)$. Otherwise $\varphi(\tau, x, u) = -\alpha x$ with $\alpha > 0$ and hence

$$\varphi(2\tau, x, u) = \varphi(\tau, \varphi(\tau, x, u), u(\tau + \cdot)) = -\alpha\,(-\alpha x) = \alpha^2 x,$$

implying $\varphi_{\mathbb{S}}(2\tau, x, u) = x$, and

$$\lambda(u,x) = \frac{1}{2\tau}\log\|\varphi(2\tau, x, u)\| = \frac{1}{2\tau}\log\alpha^2 = \frac{1}{\tau}\log\alpha \in \Sigma_{Fl}({}_{\mathbb{S}}D).$$

Analogously one argues in the case $-x \in {}_{\mathbb{S}}D$.

□

The following result describes the control sets in $\mathbb{R}^n$ under the accessibility rank condition on projective space.

Theorem 2.2.15. *Assume that the projected control system (2.5) on $\mathbb{P}^{n-1}$ satisfies the accessibility rank condition (2.13). If a control set ${}_{\mathbb{S}}D_i, i \in \{1, \ldots, k_1\}$, on $\mathbb{S}^{n-1}$ satisfies $0 \in \mathrm{int}\left(\Sigma_{Fl}({}_{\mathbb{S}}D_i)\right)$, then the cone*

$$D_i = \{\alpha x \in \mathbb{R}^n \mid \alpha > 0 \text{ and } x \in {}_{\mathbb{S}}D_i\}$$

generated by ${}_{\mathbb{S}}D_i$ is a control set with nonvoid interior in $\mathbb{R}^n \setminus \{0\}$. At most two of the D_i are invariant control sets.

Proof. By Proposition 2.2.8, the system on $\mathbb{S}^{n-1}$ satisfies the accessibility rank condition (2.13), so every point in ${}_{\mathbb{S}}D_i$ is locally accessible. Hence the first assertion of Theorem 2.2.2 is hold, if we can show that assumption (ii) in that theorem holds, then follow that D_i is a control set.

The Floquet spectrum over a control set in projective space is a bounded interval; cf. [13, Proposition 6.2.14]. By Proposition 2.2.14 the same is true for the Floquet spectrum of $\Sigma_{Fl}(_{\mathbb{S}}D_i)$.

By assumption $0 \in \text{int}(\Sigma_{Fl}(_{\mathbb{S}}D_i))$, it follows that there are points $\lambda_1 < 0$ and $\lambda_2 > 0$ such that $\lambda_1, \lambda_2 \in \Sigma_{Fl}(_{\mathbb{S}}D_i)$, so $(\lambda_1, \lambda_2) \subset \Sigma_{Fl}(_{\mathbb{S}}D_i)$.

For each $\lambda \in (\lambda_1, \lambda_2)$ there are $x_\lambda \in \mathbb{R}^n \setminus \{0\}$, a $\sigma_\lambda-$periodic control u_λ with $\pi_{\mathbb{S}}(x_\lambda) \in \text{int}(_{\mathbb{S}}D_i)$ and $\varphi_{\mathbb{S}}(\sigma_\lambda, x_\lambda, u_\lambda) = \pi_{\mathbb{S}}(x_\lambda)$. Thus there are $0 < \alpha_\lambda \in \mathbb{R}$ and $s_\lambda \in \mathbb{S}^{n-1}$ such that $\varphi(\sigma_\lambda, x_\lambda, u_\lambda) = \alpha_\lambda s_\lambda$. Fhurthemore

$$\begin{aligned}
\lambda &= \frac{1}{\sigma_\lambda} \log \|\varphi(\sigma_\lambda, x_\lambda, u_\lambda)\| \\
&= \frac{1}{\sigma_\lambda} \log \|\alpha_\lambda s_\lambda\| \\
&= \frac{1}{\sigma_\lambda} \log |\alpha_\lambda| \\
&= \frac{1}{\sigma_\lambda} \log \alpha_\lambda.
\end{aligned}$$

For $\lambda_1 < 0$ we have $\alpha_{\lambda_1} \in (0, 1)$, and for $\lambda \in (0, \lambda_2)$ we have $\alpha_\lambda \in (1, \infty)$, then for each $\alpha \in \left(0, e^{\lambda_2}\right)$ there are $s^+ \in \text{int}(_{\mathbb{S}}D_i)$, controls $u^+ \in \mathcal{U}$ and time $\sigma^+ > 0$ such that

$$\varphi(\sigma^+, s^+, u^+) = \alpha s^+. \tag{2.15}$$

This verifies assumption (ii) of Theorem 2.2.2. Therefore, every invariant control set D projects to an invariant control set on $\mathbb{S}^{n-1}$, and here there are at most two invariant control sets. □

The following example illustrates Theorem 2.2.15; cf. also [13, Examples 10.1.7 and 10.2.1], where for linear oscillators the spectral properties and the control sets in projective space are determined.

Example 2.2.16. *Consider the damped linear oscillator*

$$\ddot{x} + 3\dot{x} + (1 + u(t))x = 0 \text{ with } u(t) \in \Omega = [-\rho, \rho],$$

where $\rho \in \left(1, \frac{5}{4}\right)$ *. Hence the system equation is given by*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} + u \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1-u & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The eigenvalues of $A(u)$ *satisfy*

$$\det(\lambda I - A(u)) = \det \begin{pmatrix} \lambda & -1 \\ 1+u & \lambda+3 \end{pmatrix} = \lambda^2 + 3\lambda + 1 + u = 0,$$

and one obtains two real eigenvalues

$$\lambda_1(u) = -\frac{3}{2} - \sqrt{\frac{5}{4} - u} \text{ and } \lambda_2(u) = -\frac{3}{2} + \sqrt{\frac{5}{4} - u}$$

with corresponding eigenvectors $(x, \lambda_1(u)x)^\top$ *and* $(x, \lambda_2(u)x)^\top, x \neq 0$. *Note that* $\lambda_2(u) > 0$ *if and only if* $u \in [-\rho, -1)$. *Since for all* $u \in [-\rho, \rho]$ *one has* $\lambda_1(u) < \lambda_2(u)$, *the projected trajectories in* $\mathbb{P}^1$ *go from the eigenspace for* $\lambda_1(u)$ *to the eigenspace for* $\lambda_2(u)$. *A short computation shows that there is an open control set* ${}_{\mathbb{P}}D_1$ *and a closed invariant control set* ${}_{\mathbb{P}}D_2$ *in projective space* $\mathbb{P}^1$ *given by the projections of*

$$\left\{ \begin{bmatrix} x \\ \lambda x \end{bmatrix} \middle| \; x \neq 0, \, \lambda \in \Sigma_{Fl}({}_{\mathbb{P}}D_1) \right\}, \quad \left\{ \begin{bmatrix} x \\ \lambda x \end{bmatrix} \middle| \; x \neq 0, \, \lambda \in \overline{\Sigma_{Fl}({}_{\mathbb{P}}D_2)} \right\},$$

respectively, where by [13, Theorem 10.1.1] the Floquet spectra are

$$\Sigma_{Fl}({}_{\mathbb{P}}D_1) = \left(-\frac{3}{2} - \sqrt{\frac{5}{4} + \rho}, -\frac{3}{2} - \sqrt{\frac{5}{4} - \rho} \right) \subset (-\infty, 0),$$
$$\Sigma_{Fl}({}_{\mathbb{P}}D_2) = \left(-\frac{3}{2} + \sqrt{\frac{5}{4} - \rho}, -\frac{3}{2} + \sqrt{\frac{5}{4} + \rho} \right).$$

The control sets in $\mathbb{P}^1$ *induce four control sets on the unit circle* $\mathbb{S}^1$. *For* ${}_{\mathbb{P}}D_2$ *one obtains the two control sets* ${}_{\mathbb{S}}D_2' = -{}_{\mathbb{S}}D_2$. *Since* $u = -1 \in (-\rho, \rho)$ *and* $0 = \lambda_2(-1) \in \operatorname{int}(\Sigma_{Fl}({}_{\mathbb{P}}D_2))$, *Theorem 2.2.15 implies that there are two invariant control sets in* $\mathbb{R}^2 \setminus \{0\}$, *they are the cones*

$$D_2 = \left\{ \alpha \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; \alpha > 0, \begin{bmatrix} x \\ y \end{bmatrix} \in {}_{\mathbb{S}}D_2 \right\} \text{ and } D_2' = \left\{ \alpha \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; \alpha > 0, \begin{bmatrix} x \\ y \end{bmatrix} \in {}_{\mathbb{S}}D_2' \right\}.$$

Remark 2.2.17. *Suppose that under the assumptions of Theorem 2.2.15 an invariant control set* D_i *in* $\mathbb{R}^n \setminus \{0\}$ *exists. Then* $D_i \cup \{0\}$ *is a closed cone in* $\mathbb{R}^n$ *generated by an invariant control set on the unit sphere. If the system is not controllable, this cone does not coincide with* $\mathbb{R}^n$, *hence it is a nontrivial proper closed positively invariant cone in* $\mathbb{R}^n$. *On the other hand, Do Rocio, San Martin, and Santana [19, Section 6] present an example in* $\mathbb{R}^4$, *which is not controllable and which also does not possess a nontrivial proper closed* convex *cone* W *in* $\mathbb{R}^n$ *which is positively invariant. Here the convexity of* W *is crucial: Such cones are pointed, i.e.,* $W \cap (-W) = \{0\}$; *cf. [19, Lemma 4.1]. For an invariant control set* D_i *as in Theorem 2.2.15 the cone* $D_i \cup \{0\}$ *need not be pointed (and hence not convex), since the invariant control set may contain the real eigenspace for a complex conjugate pair of eigenvalues of* $A(u)$. *Observe that here the convex closure of this cone, which is also positively invariant, coincides with* $\mathbb{R}^n$. *An example is the three-dimensional linear oscillator in Colonius and Kliemann [13, Example10.2.3]. The* *existence of nontrivial proper closed convex positively invariant cones in $\mathbb{R}^n$ is analyzed in [19, Theorem 4.2, Theorem 4.5].*

Not all control sets on the unit sphere generate cones that are control sets in $\mathbb{R}^n \setminus \{0\}$ as indicated by the following proposition,

Proposition 2.2.18. *Suppose for the homogeneous bilinear control system (2.3) that the accessibility rank condition on $\mathbb{P}^{n-1}$ is valid. Let ${}_{\mathbb{S}}D_i$ be a control set with nonvoid interior on $\mathbb{S}^{n-1}$ and define the cone*

$$C_i = \{\alpha x \in \mathbb{R}^n \mid \alpha > 0 \text{ and } x \in {}_{\mathbb{S}}D_i\}. \tag{2.16}$$

Then the following assertions hold.

(i) If $x, y \in \operatorname{int}({}_{\mathbb{S}}D_i)$, for all $\alpha x, \beta y \in C_i$ there are $\gamma, T > 0$ and $u \in \mathcal{U}$ with $\varphi(T, \alpha x, u) = \gamma\beta y$;

(ii) If $0 \notin \Sigma_{Ly}({}_{\mathbb{S}}D_i)$ then the cone C_i is not a control set.

Proof. (i) Suppose that $x, y \in \operatorname{int}({}_{\mathbb{S}}D_i)$, then $x \in {}_{\mathbb{S}}\mathcal{O}^+(y)$, thus $\varphi_{\mathbb{S}}(T, x, u) = y$ for some $T \geq 0$ and $u \in \mathcal{U}_{pc}$. Since $\pi_{\mathbb{S}}(\beta y) = y$ one has

$$\pi_{\mathbb{S}}(\beta y) = \varphi_{\mathbb{S}}(T, \pi_{\mathbb{S}}(\alpha x), u) = \pi_{\mathbb{S}}(\varphi(T, \alpha x, u))$$

so $\varphi(T, \alpha x, u) = \gamma\beta y$ for some $\gamma \in \mathbb{R}$.

For assertion (ii), consider (x, u) with $\varphi_{\mathbb{S}}(t, \pi_{\mathbb{S}}(x), u) \in {}_{\mathbb{S}}D_i$ for all $t \geq 0$.

If $\sup \Sigma_{Ly}({}_{\mathbb{S}}D_i) < 0$, let $0 < \alpha < -\max\{\operatorname{Re}\lambda \mid \lambda \text{ an eigenvalue of } A(u)\}$. Then there is a constant $c_0 \geq 1$ such that every solution of the autonomous linear differential equation $\dot{x}(t) = A(u)x(t)$, $x(0) = x_0$, satisfies

$$\left\| e^{A(u)t}x \right\| \leq c_0 e^{-\alpha t} \|x\| \leq c_0 \quad \text{for all } t \geq 0. \tag{2.17}$$

Thus for α sufficiently bigger $\alpha x \notin {}_{\mathbb{S}}\mathcal{O}^+(x)$, which implies that C is not a control set. If $\inf \Sigma_{Ly}({}_{\mathbb{S}}D_i) > 0$ then $\|\varphi(t, x, u)\| \to \infty$. The claim follows considering the reverse time system. □

With some analogy to the definition of directional controllability in Bacciotti and Vivalda [3, Definition 1] we can introduce the following definition for the homogeneous bilinear control system (2.3).

Definition 2.2.19. *Consider the homogeneous bilinear control system (2.3). A cone $C \subset \mathbb{R}^n \setminus \{0\}$ is a directional control set, if*

(i) for all $x \in C$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in C$ for all $t \geq 0$;

(ii) for all $x, y \in C$ the line $l := \{\alpha y \mid \alpha > 0\}$ satisfies $l \cap \overline{\mathcal{O}^+(x)} \neq \varnothing$;

(iii) C is maximal with this property.

If $l' := \{\alpha y \mid \alpha \in \mathbb{R}\}$ we call 2.16 of directional control set.

By before proposition the cones C_i defined in (2.16) are the directional control sets of system (2.3).

Next we present a necessary and sufficient condition for controllability on $\mathbb{R}^n \setminus \{0\}$. The infimum and supremum Lyapunov exponents; cf. (2.15), are

$$\kappa^* = \inf_{u \in \mathcal{U}} \inf_{x \neq 0} \lambda(u, x) \text{ and } \kappa = \sup_{u \in \mathcal{U}} \sup_{x \neq 0} \lambda(u, x),$$

respectively. The following result improves Colonius and Kliemann [13, Corollary 12.2.6(iii)], where the accessibility rank condition is assumed in $\mathbb{R}^n \setminus \{0\}$.

Corollary 2.2.20. *Assume that the projected control system (2.5) on $\mathbb{P}^{n-1}$ satisfies the accessibility rank condition (2.13). Then the homogeneous bilinear control system (2.3) on $\mathbb{R}^n \setminus \{0\}$ is controllable if and only if the projected system (2.5) is controllable and $\kappa^* < 0 < \kappa$.*

Proof. Controllability on $\mathbb{R}^n \setminus \{0\}$ implies controllability on $\mathbb{P}^{n-1}$. Furthermore, asymptotic null controllability to $0 \in \mathbb{R}^n$, and hence exponential null controllability follows by [13, Corollary 12.2.3]. Thus $\kappa^* < 0$ and, by time reversal, $\kappa > 0$ follows.

Conversely, controllability on $\mathbb{P}^{n-1}$ implies by Bacciotti and Vivalda [3, Theorem 1] that ${}_{\mathbb{S}}D = \mathbb{S}^{n-1}$ is a control set. By Theorem 2.2.15, it follows that $\mathbb{R}^n \setminus \{0\}$ is a control set. This implies that for every initial point $x \neq 0$ the reachable set $\mathcal{O}^+(x)$ is dense in $\mathbb{R}^n \setminus \{0\}$, i.e., approximate controllability holds. For homogeneous bilinear control systems, Cannarsa and Sigalotti [9, Theorem 1] shows that approximate controllability implies controllability in $\mathbb{R}^n \setminus \{0\}$. This completes the proof. □

Remark 2.2.21. *The condition $\kappa^* < 0 < \kappa$ can be replaced by the requirement that $0 \in \operatorname{int}(\Sigma_{Fl}(\mathbb{P}^{n-1})) = (\kappa^*, \kappa)$. This follows, since by [13, Theorem 7.1.5(iv)] the Floquet spectrum is an interval and satisfies $\overline{\Sigma_{Fl}(\mathbb{P}^{n-1})} = [\kappa^*, \kappa]$ if $\mathbb{P}^{n-1}$ is a control set.*

Remark 2.2.22. *For control systems on semisimple Lie groups, San Martin [38, Proposition 5.6] shows the following result. Let $G \subset Sl(n, \mathbb{R})$ be a semisimple, connected, and noncompact group acting transitively on $\mathbb{R}^n \setminus \{0\}$ and let S be a semigroup with nonvoid interior in G. Then S is controllable on $\mathbb{R}^n \setminus \{0\}$ if and only if S is controllable in $\mathbb{P}^{n-1}$. In this case $0 \in (\kappa^*, \kappa) = \operatorname{int}(\Sigma_{Fl}(\mathbb{P}^{n-1}))$.*

We turn to the chain control sets in projective space. By [13, Theorem 7.1.2] every chain control set ${}_{\mathbb{P}}E_j$ contains a control set ${}_{\mathbb{P}}D_i$ with nonvoid interior. In particular, if k_0 is the number of control sets, hence the number l of chain control sets satisfies $1 \leq l \leq k_0 \leq n-1$. Furthermore, [13, Theorem 7.3.16] shows that for every chain control set ${}_{\mathbb{P}}E_j$ in $\mathbb{P}^{n-1}$

$$\{x \in \mathbb{R}^n \setminus \{0\} \mid \varphi_{\mathbb{P}}(t,x,u) \in {}_{\mathbb{P}}E_j \text{ for all } t \in \mathbb{R}\}$$

is a linear subspace and its dimension is independent of $u \in \mathcal{U}$. Consider, for a chain control set ${}_{\mathbb{P}}E_j$, the control sets ${}_{\mathbb{P}}D_1, \ldots, {}_{\mathbb{P}}D_{i_j}$ with nonvoid interior contained in ${}_{\mathbb{P}}E_j$. By [13, Theorem 7.3.16] one finds, for every $x \in \pi_{\mathbb{P}}^{-1}({}_{\mathbb{P}}E_j)$, points $x^i \in \pi_{\mathbb{P}}^{-1}\left(\overline{{}_{\mathbb{P}}D_i}\right)$ and $\alpha_i \in \mathbb{R}$ such that

$$x = \alpha_1 x^1 + \cdots + \alpha_{i_j} x^{i_j}. \tag{2.18}$$

By (1.4) the chain control sets ${}_{\mathbb{P}}E_j$ uniquely correspond to the maximal chain transitive subsets ${}_{\mathbb{P}}\mathcal{E}_j$ of the control flow on $\mathcal{U} \times \mathbb{P}^{n-1}$ via

$${}_{\mathbb{P}}\mathcal{E}_j := \left\{(u, \pi_{\mathbb{P}}x) \in \mathcal{U} \times \mathbb{P}^{n-1} \mid \pi_{\mathbb{P}}\varphi(t,x,u) \in {}_{\mathbb{P}}E_j \text{ for all } t \in \mathbb{R}\right\}. \tag{2.19}$$

Chapter 3

Control set for affine control system on $\mathbb{R}^n$

In this chapter we will study control sets for affine control systems on $\mathbb{R}^n$, we start by studying control sets around equilibrium points and later on we will study control sets for points that belong to periodic solutions. Moreover we set conditions for the system has unbounded control set with nonvoid interior. Section 3.1 we present some definitions and results about equilibrium points. Section 3.2 we proved that if the linearized system at equilibrium point is controllable, then there is a control set with a nonvoid interior containing this point. Section 3.3 we introduce the notion of hyperbolicity of an affine control system and under the rank condition we prove the existence and uniqueness of a control set with nonvoid interior for hyperbolic systems and uniformly hyperbolic. Section 3.4 we embedded an affine control system (3.1) to homogeneous bilinear control system on $\mathbb{R}^{n+1}$ and then projects the system on projective space $\mathbb{P}^n$. Then we relate controls sets of affine control system and its homogeneous part to control sets of projected system. Section 3.5 shows that all control sets with nonvoid interior are unbounded if the hyperbolicity condition specified in Definition 3.3.1 is violated. And we show that there is a single chain control set in $\mathbb{P}^n$ containing the images, by projection, of all control sets D with nonvoid interior in $\mathbb{R}^n$ and the boundary at infinity, Definition 3.4.4, of this chain control set, contains all chain control sets of the homogeneous part having nonvoid intersection with the boundary at infinity of one of the control sets D.

Recall, we consider the affine control system defined on $\mathbb{R}^n$, as in Section 1.2, given by

$$\dot{x}(t) = A(u(t))x(t) + Cu(t) + d, \tag{3.1}$$

where $u \in \Omega \subset \mathbb{R}^m$. We say that the system

$$\dot{x}(t) = A(u(t))x(t) \tag{3.2}$$

is its homogeneous part and the relation among their control sets will be studied in this chapter. For each $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_{pc}$ the solution of (3.1) and (3.2) are denoted by $\varphi(t,x,u)$ and $\varphi_{\text{hom}}(t,x,u)$, respectively, and positive orbit of x by $\mathcal{O}^+(x)$ and $\mathcal{O}^+_{\text{hom}}(x)$, analogous for negative orbit. An object of the system (3.2) will be differentiated from the respective object of the system (3.1) by attaching "hom".

3.1 Equilibria of affine control systems

For each control value $u \in \Omega$, an associated **equilibrium point** of system (3.1) is a state x_u that satisfies

$$0 = A(u)x_u + Cu + d. \tag{3.3}$$

If for $u \in \Omega$ there is a solution x_u of (3.3) and $\det A(u) = 0$, then every point in the nontrivial affine subspace $x_u + \ker A(u)$ is an equilibrium point. If there is $u \in \Omega$ with $Cu + d = 0$, then equation (3.3) always has the solution $x_u = 0$. If $\det A(u) \neq 0$, then there exists a unique equilibrium of (3.1) given by

$$x_u = -A(u)^{-1}[Cu + d]. \tag{3.4}$$

The following simple but useful result shows that for constant control u, the phase portrait of the inhomogeneous equation is obtained by shifting the origin to x_u.

Proposition 3.1.1. *Consider for constant control $u \in \Omega$ a solution $\varphi(t,x,u), t \geq 0$, of the inhomogeneous equation (3.1) and let x_u be an associated equilibrium. Then $\varphi(t,x,u) - x_u$ is a solution of the homogeneous equation $\dot{x}(t) = A(u)x(t)$ with initial value $x - x_u$.*

Proof. We compute

$$\frac{d}{dt}\left[\varphi(t,x,u) - x_u\right] = A(u)\left[\varphi(t,x,u) - x_u\right] + A(u)x_u + Cu + d = A(u)\left[\varphi(t,x,u) - x_u\right].$$

□

The proposition shows that the affine control system (3.1) is topologically conjugate to an inhomogeneous bilinear system, if there is $u^0 \in \Omega$ with $Cu^0 + d = 0$.

Proposition 3.1.2. *Suppose that there is $u^0 \in \Omega$ with $Cu^0 + d = 0$ and consider*

$$\dot{x}(t) = A(u^0)x(t) + \sum_{i=1}^{m} v_i(t)B_i x(t) + Cv(t) \text{ with } v(t) \in \Omega' := \Omega - u^0,$$

with trajectories denoted by $\psi(\cdot,x,v)$. Then the trajectories $\varphi(\cdot,x,u), u \in \mathcal{U}$, of (3.1) satisfy $\varphi(t,x,u) = \psi(t,x,v), t \in \mathbb{R}$, with controls $v(t) = u(t) - u^0, t \in \mathbb{R}$.

Proof. One computes for a solution $x(t) = \varphi(t, x, u), t \in \mathbb{R}$, of (3.1)

$$\begin{aligned}\dot{x}(t) &= Ax(t) + \sum_{i=1}^{m} u_i^0 B_i x(t) + \sum_{i=1}^{m} \left(u_i(t) - u_i^0\right) B_i x(t) + C(u(t) - u^0) + Cu^0 + d \\ &= A(u^0)x(t) + \sum_{i=1}^{m} v_i(t) B_i x(t) + Cv(t).\end{aligned}$$

□

We introduce the following notation for the set of equilibria,

$$\begin{aligned}E &= \{x \in \mathbb{R}^n \mid 0 = A(u)x + Cu + d \text{ for some } u \in \Omega\}, \\ E_0 &= \{x \in \mathbb{R}^n \mid 0 = A(u)x + Cu + d \text{ for some } u \in \operatorname{int}(\Omega)\}.\end{aligned}$$

The following discussion of systems with scalar controls follows essentially Mohler [32, Section 2.4].

Theorem 3.1.3. *Consider system (3.1) with scalar control and assume that for all $u \in \Omega$ such that $\det(A + uB) = 0$ there is no solution to equation (3.3).*

(i) Suppose that there is $u^0 \in \Omega = \mathbb{R}$ with $A + u^0 B$ nonsingular. Then there are at most $1 \leq r \leq n$ control values $v^i \in \mathbb{R}$ such that the equilibrium set is given by

$$E = \{x_u \bigm| u \in \mathbb{R} \setminus \{v^1, \dots, v^r\}\}$$

and which is the union of at most $n + 1$ smooth curves. These curves have no finite endpoints.

(ii) If Ω is a possibly unbounded interval, the equilibrium set E has at most $n + 1$ connected components.

Proof. First note that $x_u = -(A + uB)^{-1}[Cu + d]$ describes a smooth curve as long as $\det(A + uB) \neq 0$. Since $\det(A + uB)$ is a nontrivial polynomial in u of degree at most n, there are most n real roots $v^1, \dots, v^r, 0 \leq r \leq n$, of $\det(A + uB) = 0$. By our assumption the vectors $Cv^i + d$ are not in the range of $A + v^i B$.

Consider a sequence $u^k \to v^i$ for some i. If x_{u^k} remains bounded, we may assume that it converges to some $y \in \mathbb{R}^n$. For $k \to \infty$ we find

$$(A + v^i B)y = -(Cv^i + d)$$

contradicting the assumption of the theorem. It follows that x_{u^k} becomes unbounded for $k \to \infty$.

(ii) If $\Omega = [u_*, u^*], u_* < u^*$, the equilibrium set $E = \{x_u \mid u \in \Omega \setminus \{v^1, \dots, v^r\}\}$ consists of at most $n + 1$ smooth curves having no finite endpoints, with the possible exception of the equilibria corresponding to the minimum and maximum values of u in Ω, i.e., $u = u_*, u^*$. $\square$

The following example is used in Rink and Mohler [35, Example 2] and Mohler [32, Example 2 on page 32] as an example for a system that is not controllable. It illustrates the result above.

Example 3.1.4. *Consider the control system given by*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2u(t)x + y \\ x + 2u(t)y + u(t) \end{bmatrix}, \tag{3.5}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which is the inhomogeneous bilinear control system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2u & 1 \\ 1 & 2u \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = (A + uB) \begin{bmatrix} x \\ y \end{bmatrix} + Cu.$$

Note that $\det(A(u)) \neq 0$ *if* $|u| \neq \frac{1}{2}$ *and for* $u = \pm\frac{1}{2}$ *we have*

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} x + y \\ x + y \pm \frac{1}{2} \end{bmatrix}$$

thus, does not exist equilibrium point associated with $\pm\frac{1}{2}$*. The eigenvalues of* $A(u) = A + uB$ *are given by* $0 = \det(\lambda I - (A + uB)) = (\lambda - 2u)^2 - 1$*, hence* $\lambda_1(u) = 2u + 1 > \lambda_2(u) = 2u - 1$*. One finds* $\lambda_2(u) > 0$ *for* $u > \frac{1}{2}$ *and* $\lambda_1(u) < 0$ *for* $u < -\frac{1}{2}$*. For* $u \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ *one gets* $\lambda_1(u) > 0$ *and* $\lambda_2(u) < 0$*, hence the matrix* $A + uB$ *is hyperbolic.*

For every $u \in \mathbb{R}$*, the eigenspace for* $\lambda_1(u)$ *is* $\mathrm{Diag}_1 := \{(z, z)^\top \mid z \in \mathbb{R}\}$ *and the eigenspace for* $\lambda_2(u)$ *is* $\mathrm{Diag}_2 := \{(z, -z)^\top \mid z \in \mathbb{R}\}$*. For* $|u| \neq \frac{1}{2}$ *the equilibria are given by*

$$\begin{bmatrix} x_u \\ y_u \end{bmatrix} = -(A + uB)^{-1} Cu = \frac{-1}{4u^2 - 1} \begin{bmatrix} 2u & -1 \\ -1 & 2u \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \frac{u}{4u^2 - 1} \begin{bmatrix} 1 \\ -2u \end{bmatrix}.$$

Thus we see that

$$y_u = -2ux_u \text{ for } |u| \neq \frac{1}{2}. \tag{3.6}$$

The assumption of Theorem 3.1.3 is satisfied. For the asymptotes of the equilibria, equation (3.6) shows that $(x_u, y_u)^\top$ *approach the line* Diag_2 *for* $u \to \frac{1}{2}$ *and the line* Diag_1 *for* $u \to -\frac{1}{2}$*. In both cases, the equilibria become unbounded. For* $u \to \pm\infty$*, one obtains that* *the equilibria approach* $(0, -\frac{1}{2})^\top$.

This discussion shows that the set of equilibria for unbounded control u *consists of the following three connected branches*

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} x_u \\ y_u \end{bmatrix} \middle| \; u \in \left(-\frac{1}{2}, \frac{1}{2}\right) \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} x_u \\ y_u \end{bmatrix} \middle| \; u \in \left(-\infty, -\frac{1}{2}\right) \right\},$$
$$\mathcal{B}_3 = \left\{ \begin{bmatrix} x_u \\ y_u \end{bmatrix} \middle| \; u \in \left(\frac{1}{2}, \infty\right) \right\}.$$

The equilibria in $\mathcal{B}_2$ *and* $\mathcal{B}_3$ *both approach* $(0, -\frac{1}{2})^\top$ *for* $|u| \to \infty$ *(see figure 3.1). The equilibria in* $\mathcal{B}_2$ *are stable, those in* $\mathcal{B}_3$ *are totally unstable, and those in* $\mathcal{B}_1$ *yield one positive and one negative eigenvalue.*

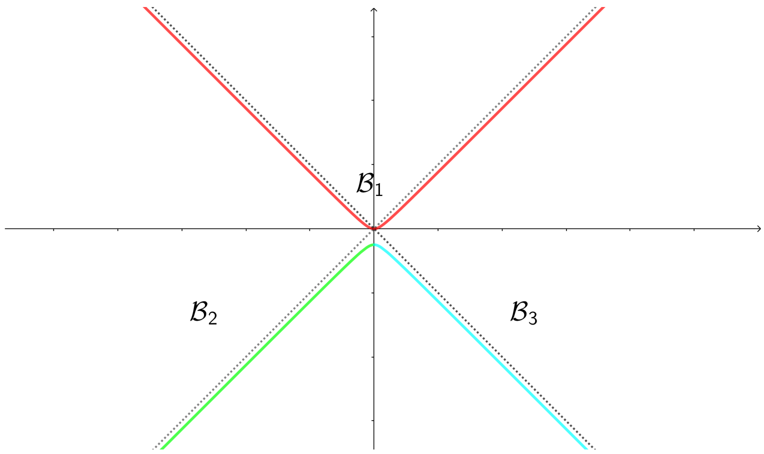

Figure 3.1: Equilibria of the system (3.5)

3.2 Control sets and equilibria of affine systems

The controllability properties near equilibria will be analyzed assuming that the linearized control systems are controllable. This yields results on the control sets around equilibria.

In order to describe the properties of the system linearized about an equilibrium, we recall the following classical result from Lee and Markus [31, Theorem 1 on p. 366].

Theorem 3.2.1. *Consider the control process in* $\mathbb{R}^n$

$$\dot{x} = f(x, u), \tag{3.7}$$

where f *is* C^1 *and suppose that* $f(0,0) = 0$ *where* 0 *is in the interior of the control range* Ω. *Then the controllable set* $\mathcal{O}^-(0)$ *is open if, with* $A = \frac{\partial f}{\partial x}(0,0)$ *and* $B = \frac{\partial f}{\partial u}(0,0)$,

$$\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n. \tag{3.8}$$

Condition (3.8) is the familiar Kalman condition for controllability of the linearized system $\dot{x} = \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial u}(0,0)u$ (without control restriction); cf. Sontag [39, Theorem 3, p. 89]. We apply this result to affine control systems and obtain that the positive orbit and the negative orbit for an equilibrium are open, if the linearized system is controllable. For this we used Lemma (1.2.2) it shows that the positive orbit of system (3.1) coincides with the negative orbit of the time reversed system.

Proposition 3.2.2. *Consider the affine control system (3.1) and let x_u be an equilibrium for a control value $u \in \operatorname{int}(\Omega)$, where the rank condition*

$$\operatorname{rank}[B'(u), A(u)B'(u), \dots, (A(u))^{n-1}B'(u)] = n \tag{3.9}$$

holds with $B'(u)$ defined by

$$B'(u) = C + [B_1 x_u, \dots, B_m x_u].$$

Then the positive orbit set $\mathcal{O}^+(x_u)$ and the negative orbit set $\mathcal{O}^-(x_u)$ are open. If $A(u) = A + \sum_{i=1}^m u_i B_i$ is invertible, then

$$B'(u) = C - \left[B_1 A(u)^{-1}(Cu+d), \dots, B_m A(u)^{-1}(Cu+d)\right].$$

Proof. First notice that Theorem 3.2.1 can be applied to arbitrary equilibria (x^0, u^0) with $u^0 \in \operatorname{int}(\Omega)$ instead of $(0,0)$. In fact, define $\tilde{f}(x,u) := f(x+x^0, u+u^0)$. Then $(0,0)$ is an equilibrium of

$$\dot{x}(t) = \tilde{f}(x(t), u(t)) \text{ with control range } \Omega - u^0, \tag{3.10}$$

and the control value $u = 0$ belongs to $\operatorname{int}(\Omega - u^0)$. The solutions $\psi(t,0,u), t \geq 0$, of (3.10) are given by $\varphi(t, x^0, u+u^0) - x^0$, since $\varphi(0, x^0, u+u^0) - x^0 = 0$ and

$$\begin{aligned}\frac{d}{dt}\left[\varphi(t,x^0,u+u^0) - x^0\right] &= f(\varphi(t,x^0,u+u^0), u(t)+u^0)\\ &= f(\varphi(t,x^0,u+u^0) + x_0 - x_0, u(t)+u^0)\\ &= \tilde{f}(\varphi(t,x^0,u+u^0) - x^0, u(t)).\end{aligned}$$

Hence $\mathcal{O}^-(x^0)$ for the affine control system (3.7) coincides with the controllable set $\tilde{\mathcal{O}}^-(0)$ of (3.10). The rank condition (3.8) for (3.10) involves

$$A = \frac{\partial \tilde{f}}{\partial x}(0,0) = \frac{\partial f}{\partial x}(x^0,u^0), B = \frac{\partial \tilde{f}}{\partial u}(0,0) = \frac{\partial f}{\partial u}(x^0,u^0).$$

For system (3.1) $f(x,u) = A(u)x + Cu + d$ and for an equilibrium x_u we find $\frac{\partial f}{\partial x}(x_u,u) =$

$A(u)$ and

$$\frac{\partial f}{\partial u}(x_u, u) = C + \frac{\partial}{\partial u}\sum_{i=1}^{m} u_i B_i x_u = C + [B_1 x_u, \ldots, B_m x_u] .$$

By (3.9) the rank condition (3.8) is satisfied. Applying Theorem 3.2.1 we conclude that the controllable set $\mathcal{O}^-(x_u)$ is open. By time reversal; cf. Lemma 1.2.2, also the reachable set $\mathcal{O}^+(x_u)$ is open.

If $A(u)$ is invertible, the formula for $B'(u)$ follows from (3.4). □

The following proposition shows that the controllability rank condition (3.9) holds generically for controls $u \in \mathbb{R}^m$ if it holds in some u^0.

Proposition 3.2.3. *Assume that $A(u)$ is invertible for all $u \in \mathbb{R}^m$ and that the rank condition (3.9) holds for some $u^0 \in \mathbb{R}^m$. Then (3.9) holds for all u in an open and dense subset of $\mathbb{R}^m$.*

Proof. Define

$$B''(u) := \det A(u)C - [B_1 \mathrm{Adj}(A(u))(Cu + d), \ldots, B_m \mathrm{Adj}(A(u))(Cu + d)] ,$$

where $\mathrm{Adj}(A(u)) = \det A(u)\,(A(u))^{-1}$. As $A(u)$ is invertible by Propositon 3.2.2

$$B'(u) = C - \left[B_1 A(u)^{-1}(Cu + d), \ldots, B_m A(u)^{-1}(Cu + d)\right] ,$$

so $B''(u) = \det A(u)B'(u)$, in this case

$$\begin{aligned} & \mathrm{rank}[B'(u), A(u)B'(u), \ldots, (A(u))^{n-1} B'(u)] \\ = \; & \mathrm{rank}[\det A(u)B'(u), \det A(u)A(u)B'(u), \ldots, \det A(u)\,(A(u))^{n-1} B'(u)] \\ = \; & \mathrm{rank}[B''(u), A(u)B''(u), \ldots, (A(u))^{n-1} B''(u)]. \end{aligned}$$

Thus the rank condition (3.9) holds if and only if

$$\mathrm{rank}[B''(u), A(u)B''(u), \ldots, (A(u))^{n-1} B''(u)] = n. \tag{3.11}$$

The entries of the matrix in (3.11) are polynomial in the variables $u_1, \ldots, u_m$. Using the assumption one finds that the set of $u \in \mathbb{R}^m$ that not satisfy (3.11) is contained in a proper algebraic variety; the complement of such a set is open and dense in $\mathbb{R}^m$ (this follows in the same way as the genericity of the controllability rank condition (3.8); cf. Sontag [39, Proposition 3.3.12]). □

Remark 3.2.4. *For a system of the form (3.1) with scalar control, the assumptions of Proposition 3.2.3 imply that there are at most finitely many u such that the rank condition*

(3.9) *is not satisfied. This follows taking into account that for scalar* u *the entries of the matrix in* (3.11) *are polynomial in the scalar variable* u*, hence there are at most finitely many zeros.*

A consequence of Proposition 3.2.2 is the following first result on control sets.

Proposition 3.2.5. *Consider the affine control system* (3.1) *and assume that the rank condition* (3.9) *is satisfied for some* $u \in \text{int}(\Omega)$*. Then the set* $D = \mathcal{O}^-(x_u) \cap \overline{\mathcal{O}^+(x_u)}$ *is a control set of system* (3.1) *containing the equilibrium* x_u *in the interior.*

Proof. By Proposition 3.2.2 the sets $\mathcal{O}^-(x_u)$ and $\mathcal{O}^+(x_u)$ are open neighborhoods of x_u, hence it follows that x_u is in the interior of the set $D_0 := \mathcal{O}^-(x_u) \cap \overline{\mathcal{O}^+(x_u)}$.

Let $x \in D_0$. Then $x_u \in \mathcal{O}^+(x)$ and therefore $\mathcal{O}^+(x_u) \subset \mathcal{O}^+(x)$ and as $D_0 \subset \overline{\mathcal{O}^+(x_u)}$, it follows that $D_0 \subset \overline{\mathcal{O}^+(x)}$. Next we show that there is a control $v \in \mathcal{U}$ with $\varphi(t, x, v) \in D_0$ for all $t \geq 0$. Since $x \in \mathcal{O}^-(x_u)$ there are $T > 0$ and $v_1 \in \mathcal{U}$ such that $\varphi(T, x, v_1) = x_u$ and $\varphi(t, x, v_1) \in \mathcal{O}^-(x_u)$ for all $t \in [0, T]$. Furthermore, $\varphi(t, x, v) \in \mathcal{O}^+(x)$ and $x \in \overline{\mathcal{O}^+(x_u)}$, and hence continuous dependence on the initial value shows that $\varphi(t, x, v) \in \overline{\mathcal{O}^+(x_u)}$ for all $t \in [0, T]$. Now the control function

$$w(t) := \begin{cases} v(t) & \text{for} \quad t \in [0, T] \\ u(t) & \text{for} \quad t > T \end{cases}$$

yields $\varphi(t, x, w) \in D_0$ for all $t \geq 0$. We have shown that D_0 satisfies properties (i) and (ii) in Definition 1.1.7. Hence it is contained in a maximal set D with these properties, i.e., a control set, obtained as the union of all sets satisfying properties (i) and (ii) and containing D_0.

Let us show that $D_0 = D$. By the definition of control sets and $x_u \in D$, the inclusion $D \subset \overline{\mathcal{O}^+(x_u)}$ holds and for $x \in D$ one has $x_u \in \overline{\mathcal{O}^+(x)}$. Using that $\mathcal{O}^-(x_u)$ is a neighborhood of x_u this implies that there are $T > 0$ and a control $u \in \mathcal{U}$ with $\varphi(T, x, u) \in \mathcal{O}^-(x_u)$, and hence $x \in \mathcal{O}^-(x_u)$. This shows that $D \subset \mathcal{O}^-(x_u) \cap \overline{\mathcal{O}^+(x_u)} = D_0$ and hence equality holds concluding the proof that D_0 is a control set. □

Next we show that every connected subset of the set E_0 of equilibria is contained in a single control set, if the systems linearized about the equilibria are controllable.

Theorem 3.2.6. *Let* $\mathcal{C} \subset \{x_u \mid u \in \text{int}(\Omega)\} = E_0$ *be a pathwise connected subset of the set of equilibria of system* (3.1) *and assume that for every equilibrium* x_u *in* $\mathcal{C}$ *the control* u *satisfies the rank condition* (3.9)*. Then there exists a control set* D *containing* $\mathcal{C}$ *in the interior and* $D = \mathcal{O}^-(x_u) \cap \overline{\mathcal{O}^+(x_u)}$ *for every* $x_u \in \mathcal{C}$*.*

Proof. By Proposition 3.2.5 every equilibrium $x_u \in \mathcal{C}$ is contained in the interior of a control set.

Consider two points x_u and x_v in $\mathcal{C}$. Then $x_v \in \mathcal{O}^+(x_u)$. In fact, consider a continuous path from x_u to x_v in $\mathcal{C}$, say $h : [0,1] \to \mathcal{C}$ with $h(0) = x_u$ and $h(1) = x_v$. Suppose that there is $h(T) \notin \mathcal{O}^+(x_u)$ for some $T > 0$, so there is a point $x_w \in \partial(\mathcal{O}^+(x_u)) \cap \mathcal{C}$, by Proposition 3.2.2 $\mathcal{O}^-(x_w)$ is a neighborhood of x_w and there is $x_a \in \mathcal{O}^+(x_u) \cap \mathcal{O}^-(x_w)$. Furthermore, $x_w \in \mathcal{O}^+(x_a) \subset \mathcal{O}^+(x_u)$, which is a contradiction. Thus one can steer the system from any point $x_u \in \mathcal{C}$ to any other point $x_v \in \mathcal{C}$. It follows that $\mathcal{C}$ is contained in a single control set D. The same arguments show that, in fact, $\mathcal{C}$ is contained in the interior of D. □

Remark 3.2.7. *For scalar control, Theorem 3.1.3 shows that there are at most $n+1$ connected components of the set E of equilibria, which consists of at most $n+1$ smooth curves. Thus also E_0 consists of at most $n+1$ smooth curves which, naturally, are pathwise connected. Hence, under the assumptions of Theorem 3.2.6, there are at most $n+1$ control sets containing an equilibrium in the interior.*

Example 3.2.8. *Let $0 < \rho \in \mathbb{R}$ and $\Omega = [-\rho, \rho]$. Consider the affine control system defined on $\mathbb{R}^2$ by*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} + u \begin{pmatrix} c \\ d \end{pmatrix} \tag{3.12}$$

with $a, b, c, d > 0$. First we will determine the equilibria point of affine control system (3.31). For $u = 0$ there is no equilibrium point associated with u, and for $u \neq 0$ the equilibrium point of the system associated with u is the point $\left(-a - \frac{c}{u}, b + \frac{d}{u}\right)^\top$. In fact

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \begin{pmatrix} x_u \\ y_u \end{pmatrix} + u \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$$
$$\begin{pmatrix} x_u \\ y_u \end{pmatrix} = \begin{pmatrix} -a - \frac{c}{u} \\ b + \frac{d}{u} \end{pmatrix}.$$

Note that the set of the equilibria point has two branches, $\mathcal{B}_1 = \left\{ (x_u, y_u)^\top \middle| \ u \in [-\rho, 0) \right\}$ and $\mathcal{B}_2 = \left\{ (x_u, y_u)^\top \middle| \ u \in (0, \rho] \right\}$.

Its follows from $c, d > 0$ that

$$\lim_{u \to 0^+} -a - \tfrac{c}{u} = -\infty, \quad \lim_{u \to 0^-} -a - \tfrac{c}{u} = \infty$$
$$\lim_{u \to 0^+} b + \tfrac{d}{u} = \infty, \quad \lim_{u \to 0^-} b + \tfrac{d}{u} = -\infty$$

and $v \in [-\rho, 0)$ implies that for $v \to 0^-$

$$-a + \frac{c}{\rho} < -a - \frac{c}{v} = x_v \to \infty \ , \ b - \frac{d}{\rho} > b + \frac{d}{v} = y_v \to -\infty,$$

and $v \in (0, \rho]$ *implies that for* $v \to 0^+$

$$-a - \frac{c}{\rho} > -a - \frac{c}{v} = x_v \to -\infty \;, b + \frac{d}{\rho} < b + \frac{d}{v} = y_v \to \infty.$$

In figure 3.2, the red half-lines represent the set of equilibrium points.

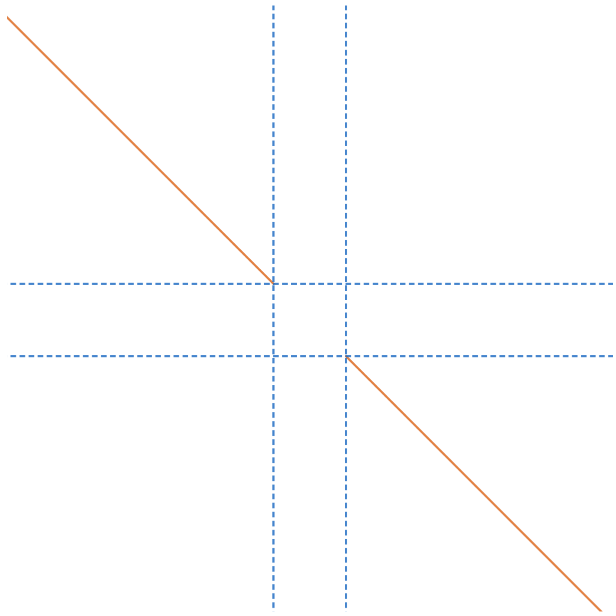

Figure 3.2: Equilibria of system (3.31)

We will check the rank condition (3.9) for each equilibrium:

$$\begin{pmatrix} a\left(1-\frac{1}{u}\right) & ua\left(1-\frac{1}{u}\right) \\ b\left(1-\frac{1}{u}\right) & -ub\left(1-\frac{1}{u}\right) \end{pmatrix} = \left(1-\frac{1}{u}\right)\begin{pmatrix} a & ua \\ b & -ub \end{pmatrix}.$$

As $a, b \neq 0$ *then the rank of the matrix above is* 2.

In this case the sets $\mathcal{B}_1' = \left\{ (x_u, y_u)^\top \middle| \; u \in (-\rho, 0) \right\}$ *and* $\mathcal{B}_2' = \left\{ (x_u, y_u)^\top \middle| \; u \in (0, \rho) \right\}$ *are contained in the interior of some control set.*

Note that for all $u \neq 0$ *the matrix* $A(u)$ *has one positive eigenvalue and one negative eigenvalue.*

Let $\mathcal{C}_1$ *the control set of the system (3.31) containing* $\mathcal{B}_1'$. *We will prove that:*

$$\mathcal{C}_1 = \left\{ (x, y) | \; x \geq -a + \frac{c}{\rho}, \; y \leq b - \frac{d}{\rho} \right\}. \tag{3.13}$$

For every equilibrium (x_u, y_u) *with* $u \in (-\rho, 0)$ *the intersection* $\mathcal{O}^-(x_u, y_u) \cap \overline{\mathcal{O}^+(x_u, y_u)}$ *is a control set. In addition, as* $\mathcal{B}_1'$ *is connected and* $\mathcal{O}^+(x_u, y_u)$ *and* $\mathcal{O}^-(x_u, y_u)$ *are open sets, follow that* $\mathcal{O}^+(x_u, y_u) = \mathcal{O}^+(x_v, y_v)$ *and* $\mathcal{O}^-(x_u, y_u) = \mathcal{O}^-(x_v, y_v)$, *for all* $u, v \in (-\rho, 0)$.

Step 1. *For each* (x_u, y_u)*, with* $u \in (-\rho, 0)$*,* $\mathcal{O}^+(x_u, y_u) \subset \left\{(x,y) |\ x \geq -a + \frac{c}{\rho}\right\}$ *and* $\mathcal{O}^-(x_u, y_u) \subset \left\{(x,y) |\ y \leq b - \frac{d}{\rho}\right\}$.

The solution starting in (x, y) *with constant control* u *is*

$$\varphi(t, (x,y), u) = \left(e^{tu}(x - x_u) + x_u, e^{-tu}(y - y_u) + y_u\right)$$

for $u \neq 0$ *and*

$$\varphi(t, (x,y), 0) = (ta + x, tb + y).$$

Remark that $x_u \leq x_\rho = -a - \frac{c}{\rho} < x_{-\rho} = -a + \frac{c}{\rho} < x_v$*, for all* $v < 0 < u$*.*

Let $x \geq -a + \frac{c}{\rho}$ *then:*

i) For $v \in (0, \rho]$*, so* $-a + \frac{c}{\rho} \leq x \leq e^{tv}(x - x_v) + x_v$*.*

ii) For $v < 0$ *and* $x < x_v$*, then* $-a + \frac{c}{\rho} < x \leq e^{tv}(x - x_v) + x_v < x_v$*.*

iii) For $v < 0$ *and* $x_v < x$*, so* $-a + \frac{c}{\rho} \leq x_v < e^{tv}(x - x_v) + x_v \leq x$*.*

Therefore, as $x_u \geq -a + \frac{c}{\rho}$ *for all* $u \in [-\rho, 0)$ *it follows that*

$$\mathcal{O}^+(x_u, y_u) \subset \left\{(x,y) |\ x \geq -a + \frac{c}{\rho}\right\} \text{ for all } u < 0.$$

Remark that $y_u \geq y_\rho = b + \frac{d}{\rho} \geq y_{-\rho} = b - \frac{d}{\rho} \geq y_v$*, for all* $v < 0 < u$*.*

Now, if $y > b - \frac{d}{\rho}$ *then:*

iv) For $v < 0$ *we have* $e^{-tv}(y - y_v) + y_v \geq y > b - \frac{d}{\rho}$*.*

v) For $v > 0$ *and* $y < y_v$ *we have* $b - \frac{d}{\rho} < y \leq e^{-tv}(y - y_v) + y_v$*.*

vi) For $v > 0$ *and* $y > y_v$ *we have* $b - \frac{d}{\rho} < y_v < e^{-tv}(y - y_v) + y_v$*.*

As $y_u \leq b - \frac{d}{\rho}$ *for all* $u \in [-\rho, 0)$ *we have that if* $(x,y) \in \mathbb{R}^2$ *and* $y > b - \frac{d}{\rho}$ *then* $(x_u, y_u) \notin \mathcal{O}^+(x,y)$*. Therefore*

$$\mathcal{O}^-(x_u, y_u) \subset \left\{(x,y) |\ y \leq b - \frac{d}{\rho}\right\} \text{ for } u < 0.$$

This ensures that $\mathcal{C}_1 \subset \left\{(x,y) |\ x \geq -a + \frac{c}{\rho},\ y \leq b - \frac{d}{\rho}\right\}$.

Step 2. *We have that* $\left\{(x,y) |\ x > -a + \frac{c}{\rho},\ y \leq b - \frac{d}{\rho}\right\} \subset \mathcal{O}^-(x_u, y_u)$ *for all* $u < 0$*.*

In fact, for all $(x,y) \in \left\{(x,y) |\ x \geq -a + \frac{c}{\rho},\ y \leq b - \frac{d}{\rho}\right\}$ *there are* $w, v \in [-\rho, 0)$ *such that*

$$(x,y) = (x_w, y_v).$$

Thus

$$\begin{aligned}\varphi(t, (x_w, y_v), v) &= \left(e^{tv}(x_w - x_v) + x_v, e^{-tv}(y_v - y_v) + y_v\right)\\ &= \left(e^{tv}(x_w - x_v) + x_v, y_v\right)\end{aligned}$$

converges to (x_v, y_v) *when* t *converge to infinite and* $\mathcal{O}^-(x_v, y_v)$ *is open, then* $(x_w, y_v) \in \mathcal{O}^-(x_v, y_v)$.

Step 3. *Now we will check that* $\left\{(x,y)| \ x \geq -a + \frac{c}{\rho}, \ y \leq b - \frac{d}{\rho}\right\} \subset \overline{\mathcal{O}^+(x_u, y_u)}$ *for all* $u < 0$.

a) If $y_v < y_w$*, then* $v > w \geq -\rho$ *and* $y_v < y_w \leq y_{-\rho}$*, as*

$$\lim_{t\to\infty} e^{t\rho}(y_w - y_{-\rho}) + y_{-\rho} = -\infty,$$

then there is $t_1 > 0$ *such that* $e^{t_1\rho}(y_w - y_{-\rho}) + y_{-\rho} = y_v$.

Let $x_1 = e^{-t_1\rho}(x_w - x_{-\rho}) + x_{-\rho}$*, as* $x_v > x_w \geq x_{-\rho}$ *and* $e^{tw} \leq 1$ *it follows that* $e^{-t\rho}(x_w - x_{-\rho}) + x_{-\rho} \leq x_w$*, so* $x_1 \leq x_w$*. Therefore, there is* $t_2 > 0$ *such that* $\varphi(t_2, (x_1, y_v), v) = (x_w, y_v)$*, because*

$$\begin{aligned}\varphi(t, (x_1, y_v), v) &= \left(e^{tv}(x_1 - x_v) + x_v, e^{-tv}(y_v - y_v) + y_v\right)\\ &= \left(e^{tv}(x_1 - x_v) + x_v, y_v\right),\end{aligned}$$

$x_1 \leq e^{tv}(x_1 - x_v) + x_v < x_v$ *and* $\lim_{t\to\infty}(e^{tv}(x_1 - x_v) + x_v, y_v) = (x_v, y_v)$.

b) If $y_v > y_w$ *we choose* $u > w$*, thus* $y_w < y_v < y_u$ *and*

$$\varphi(t, (x_u, y_u), \rho) = \left(e^{t\rho}(x_u - x_\rho) + x_\rho, e^{-t\rho}(y_u - y_\rho) + y_\rho\right).$$

As $\lim_{t\to\infty} e^{-t\rho}(y_u - y_\rho) + y_\rho = y_\rho$ *there is* $t_1 > 0$ *such that* $e^{-t_1\rho}(y_u - y_\rho) + y_\rho = y_v$.
Let $x_1 = e^{t_1\rho}(x_u - x_\rho) + x_\rho$*. Considering the solution* $\varphi(t, (x_1, y_v), v)$ *we have that*

$$\varphi(t, (x_1, y_v), w) = \left(e^{tv}(x_1 - x_v) + x_v, y_v\right),$$

as $\lim_{t\to\infty}(e^{tv}(x_1 - x_v) + x_v, y_v) = (x_v, y_v)$ *there is* $t_2 > 0$ *such that* $\varphi(t_2, (x_1, y_v), v) = (x_w, y_v)$*. Therefore,* $(x, y) \in \overline{\mathcal{O}^+(x_u, y_u)}$.

This complete the proof that $\mathcal{C}_1 = \left\{(x,y)| \ x \geq -a + \frac{c}{\rho}, \ y \leq b - \frac{d}{\rho}\right\}$*. In an analogous way we can show that the control set containing* $\mathcal{B}_2'$ *is*

$$\mathcal{C}_2 = \left\{(x,y)| \ x \leq -a - \frac{c}{\rho}, \ y \leq b - \frac{d}{\rho}\right\}. \tag{3.14}$$

In the figure 3.3, the blue region represent the control sets (3.13) and (3.14).

Here we determine two invariant control sets, but we cannot claim that these are the only control sets of this system.

In the rest of this section, we relate the controllability properties of system (3.1) to spectral properties of the matrices $A(u), u \in \Omega$.

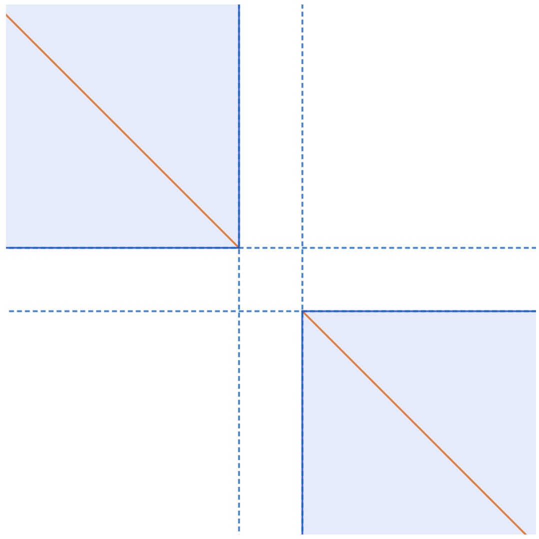

Figure 3.3: Control sets of the system (3.2.8)

Lemma 3.2.9. *Consider the affine control system (3.1) and suppose that x_u is an equilibrium for a control value $u \in \operatorname{int}(\Omega)$ satisfying the rank condition (3.9).*

(i) If every eigenvalue of $A(u)$ has negative real part, it follows that $\mathcal{O}^-(x_u) = \mathbb{R}^n$.

(ii) If every eigenvalue of $A(u)$ has positive real part, it follows that $\mathcal{O}^+(x_u) = \mathbb{R}^n$.

Proof. By Proposition 3.2.2 the rank condition (3.9) implies that $\mathcal{O}^-(x_u)$ and $\mathcal{O}^+(x_u)$ are open.

(i) Let $0 < \alpha < -\max\{\operatorname{Re}\lambda \mid \lambda \text{ an eigenvalue of } A(u)\}$. Then there is a constant $c_0 \geq 1$ such that every solution of the autonomous linear differential equation $\dot{x}(t) = A(u(t))x(t)$, $x(0) = x_0$, satisfies

$$\left\| e^{A(u(t))} x_0 \right\| \leq c_0 e^{-\alpha t} \left\| x_0 \right\| \text{ for all } t \geq 0. \tag{3.15}$$

The variation-of-constants formula applied for $x \in \mathbb{R}^n$ and x_u shows that

$$\begin{aligned} &\varphi(t,x,u) - x_u \\ &= e^{A(u)t}x + \int_0^t e^{A(u)(t-s)}[Cu+d]ds - e^{A(u)t}x_u - \int_0^t e^{A(u)(t-s)}[Cu+d]ds \\ &= e^{A(u)t}\left(x - x_u\right). \end{aligned}$$

Thus (3.15) implies

$$\left\|\varphi(t,x,u) - x_u\right\| \leq c_0 e^{-\alpha t} \left\|x - x_u\right\| \to 0 \text{ for } t \to \infty.$$

Since $\mathcal{O}^-(x_u)$ is a neighborhood of x_u, there exists $T > 0$ such that $\varphi(T,x,u) \in \mathcal{O}^-(x_u)$. Thus $x \in \mathcal{O}^-\left(\varphi(T,x,u)\right) \subset \mathcal{O}^-(x_u)$ and $\mathbb{R}^n = \mathcal{O}^-(x_u)$ follows.

(ii) For the system $\dot{x}(t) = -A(u)x - Cu - d$, every eigenvalue of $-A(u)$ has negative real part. By (i) and time reversal, Lemma 1.2.2, the assertion follows. □

Remark 3.2.10. *An easy consequence of this lemma is that the system is controllable if there are $u, v \in \Omega$ with equilibria x_u, x_v in the same pathwise connected subset of E_0 such that every eigenvalue of $A(u)$ has negative real part and every eigenvalue of $A(v)$ has positive real part; cf. Mohler [32, Main Result, p. 28] for the special case of inhomogeneous bilinear systems of the form*

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t)B_i x(t) + Cu(t).$$

The following corollary to Theorem 3.2.6 shows that there is a control set around the set of equilibria for uniformly hyperbolic matrices $A(u), u \in \Omega$.

Corollary 3.2.11. *Consider an affine control system of the form (3.1) and assume that:*

(i) the control range $\Omega = \overline{\operatorname{int}(\Omega)}$ is compact and $\operatorname{int}(\Omega)$ is pathwise connected;

(ii) the matrices $A(u)$ are uniformly hyperbolic in the following sense: There is k with $0 \leq k \leq n$ such that for all $u \in \Omega$ there are k eigenvalues with $\operatorname{Re}\lambda_1(u), \ldots, \operatorname{Re}\lambda_k(u) < 0$ and $n-k$ eigenvalues with $\operatorname{Re}\lambda_{k+1}(u), \ldots, \operatorname{Re}\lambda_n(u) > 0$;

(iii) every $u \in \operatorname{int}(\Omega)$ satisfies the rank condition (3.9).

Then the set $E = \overline{E_0}$ of equilibria is compact and connected, the set E_0 is pathwise connected, and there exists a control set D with $E_0 \subset \operatorname{int}(D)$.

Proof. First observe that all matrices $A(u)$, $u \in \Omega$, are invertible, since 0 is not an eigenvalue. Thus the function $x_\cdot : \Omega \longrightarrow \mathbb{R}^n$, given by $u \mapsto x_u$ is continuous, so the set $E = \{x_u \mid u \in \Omega\}$ of equilibria is compact and E_0 is pathwise connected. By Theorem 3.2.6 there exists a control set containing E_0 in the interior. Since pathwise connected sets are connected, the set $\operatorname{int}(\Omega)$ is connected, which implies that also $\Omega = \overline{\operatorname{int}(\Omega)}$ is connected; cf. Engelking [22, Corollary 6.1.11]. It also follows that the set $E = \overline{E_0}$ is connected. □

If condition (ii) of Corollary 3.2.11 holds with $k = 0$ or $k = n$, i.e., if all matrices $A(u)$ are stable or all are totally unstable, the rank condition (iii) for the linearized systems can be weakened.

Corollary 3.2.12. *Suppose that the assumption (i) of Corollary 3.2.11 is satisfied and assume that there are at most finitely many points in $\operatorname{int}(\Omega)$ such that the rank condition (3.9) is no valid.*

(i) *If for all* $u \in \operatorname{int}(\Omega)$ *all eigenvalues of* $A(u)$ *have negative real parts, then there exists a closed control set* D *with* $E_0 \subset \operatorname{int}(D)$.

(ii) *If for all* $u \in \operatorname{int}(\Omega)$ *all eigenvalues of* $A(u)$ *have positive real parts, then there exists a control set* D *with* $E_0 \subset \overline{D}$.

Proof. As in Corollary 3.2.11 (i) it follows that the set E_0 of equilibria is pathwise connected. Consider equilibria $x_u, x_v \in E_0$ with $u, v \in \operatorname{int}(\Omega)$ and suppose that x_u satisfies condition (3.9). Hence there is a control set D_u containing x_u in the interior. We use a construction similar to the one in the proof of Theorem 3.2.6: There is a continuous map $h : [0,1] \to E_0$ with $h(0) = x_u$ and $h(1) = x_v$. Let

$$\tau := \sup\{s \in [0,1] \mid \forall s' \in [0,s], h(s') \in D_u\}.$$

Observe that $\tau > 0$, since $x_u \in \operatorname{int}(D_u)$. If $\tau < 1$, then $y := h(\tau) \in \partial D_u$ and $y = x_w$ is an equilibrium for some $w \in \operatorname{int}(\Omega)$. If w satisfies (3.9), then by Proposition 3.2.5 x_w is in the interior of a control set, contradicting the choice of τ. It remains to discuss the case where w violates (3.9).

(i) Since all eigenvalues of $A(u)$ have negative real parts, Lemma 3.2.9 (i) implies that $x_w \in \mathcal{O}^-(x_u) = \mathbb{R}^n$. Hence there are $T \geq 0$ and $u^0 \in \mathcal{U}_{pc}$ such that $\varphi(T, x_w, u^0) = x_u$, that is, one can steer x_w (in finite time) into the interior of D_u, and by continuous dependence on the initial value, this holds for all x in a neighborhood $N(x_w)$. Note that $x_w \in \overline{D_u} \cap \partial D_u$. Since there are only finitely many points that not satisfy (3.9), all points $h(s'')$ with $s'' \in (\tau, \tau+\varepsilon)$ for some $\varepsilon > 0$ satisfy (3.9) and hence they are in a single control set D' and hence $x_w \in \overline{D'}$. Then all points in the nonvoid intersection $N(x_w) \cap D'$ can be steered into D_u. The same arguments show that one can steer points in D_u into D', hence $D' = D_u$. This contradicts the choice of τ. It follows that $\tau = 1$ and $x_v \in \overline{D_u}$. Using $x_v \in \mathcal{O}^-(x_u) = \mathbb{R}^n$ and $D_u = \overline{\mathcal{O}^+(x_u)} \cap \mathcal{O}^-(x_u) = \overline{\mathcal{O}^+(x_u)}$ one sees that $x_v \in D_u$. We conclude that all equilibria in E_0 are contained in the interior of a single closed control set.

(ii) Since all eigenvalues of $A(u)$ have positive real parts, Lemma 3.2.9 (ii) implies that $x_w \in \mathcal{O}^+(x_u) = \mathbb{R}^n$. This shows that x_w can be reached from $x_u \in \operatorname{int}(D_u)$. Notice that the continuous dependence on the initial value shows that all points in a neighborhood $N(x_w)$ of x_w can be reached from the interior of D_u. Since there are only finitely many points violating (3.9), all points $h(s'')$ with $s'' \in (\tau, \tau+\varepsilon)$ for some $\varepsilon > 0$ are in a single control set D' and $x_w \in \overline{D'}$. Then all points in the nonvoid intersection $N(x_w) \cap D'$ can be reached from the interior of D_u. The same arguments show that some point in $\operatorname{int}(D_u)$ can be reached from D', hence $D' = D_u$. This contradicts the choice of τ. It follows that $\tau = 1$ and $x_v \in \overline{D_u}$. We conclude that all equilibria in E_0 are contained in the closure of a single control set. □

Remark 3.2.13. *Remark 3.2.4 shows for an affine control system of the form (3.1) with scalar control satisfying the assumptions of Proposition 3.2.3 that there are at most finitely many points u where the rank condition (3.9) is not satisfied.*

In the next example we will consider $A(u)$, a 2×2 diagonal matrix and under the assumption that $\det(A(u)) \neq 0$, for all u. We will given conditions to guarantee that E_0 is contained in a unique control set.

Example 3.2.14. *Consider the affine control system defined on* $\mathbb{R}^2$ *by*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \left(\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + u \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} + u \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{3.16}$$

where $\Omega = [u_*, u^*]$, *with* $u_* < 0 < u^*$. *Assume that*

$$\lambda_1(u) = a_{11} + ub_{11}, \lambda_2(u) = a_{22} + ub_{22} \neq 0, \text{ for all } u \in \Omega.$$

For each $u \in \Omega$ *every equilibrium satisfies*

$$\begin{bmatrix} x_u \\ y_u \end{bmatrix} = - \begin{bmatrix} \frac{1}{a_{11}+ub_{11}} & 0 \\ 0 & \frac{1}{a_{22}+ub_{22}} \end{bmatrix} \begin{bmatrix} uc_1 + d_1 \\ uc_2 + d_2 \end{bmatrix}.$$

Next we check if the following rank condition is satisfied,

$$\text{rank}[B'(v), A(v)B'(v)] = 2.$$

Here

$$B'(v) = B \begin{bmatrix} x_v \\ y_v \end{bmatrix} + C = \begin{bmatrix} b_{11}x_v + c_1 \\ b_{22}y_v + c_2 \end{bmatrix} = \begin{bmatrix} -b_{11}\frac{vc_1+d_1}{a_{11}+b_{11}v} + c_1 \\ -b_{22}\frac{vc_2+d_2}{a_{22}+b_{22}v} + c_2 \end{bmatrix}.$$

We recall that

$$A(v) = A + vB = \begin{bmatrix} \frac{1}{a_{11}+b_{11}v} & 0 \\ 0 & \frac{1}{a_{22}+b_{22}v} \end{bmatrix},$$

hence

$$[B'(v), A(v)B'(v)] = \begin{bmatrix} -b_{11}\frac{vc_1+d_1}{a_{11}+b_{11}v} + c_1 & \frac{1}{a_{11}+b_{11}v}\left(-b_{11}\frac{vc_1+d_1}{a_{11}+b_{11}v} + c_1\right) \\ -b_{22}\frac{vc_2+d_2}{a_{22}+b_{22}v} + c_2 & \frac{1}{a_{22}+b_{22}v}\left(-b_{22}\frac{vc_2+d_2}{a_{22}+b_{22}v} + c_2\right) \end{bmatrix}.$$

The rank of this matrix is equal to 2 if

$$c_1 \neq b_{11}\frac{vc_1 + d_1}{a_{11} + b_{11}v} \text{ and } c_2 \neq b_{22}\frac{vc_2 + d_2}{a_{22} + b_{22}v} \tag{3.17}$$

and

$$a_{11} + b_{11}v \neq a_{22} + b_{22}v. \tag{3.18}$$

The latter condition holds for all $v \in \mathrm{int}(\Omega)$ *except possibly for*

$$v = \frac{a_{11} - a_{22}}{b_{22} - b_{11}}.$$

Similarly, also (3.17) holds for all $v \in \mathrm{int}(\Omega)$ *except possibly two points.*

If the conditions (3.17) and (3.18) are holds, then by Corollary 3.2.11 we conclude that there is a control set D *containing the equilibria* $\{(x_u, y_u) | \ u \in (u_*, u^*)\}$ *in its interior. Moreover, if* $\lambda_1(u), \lambda_2(u) < 0$ *or* $\lambda_1(u), \lambda_2(u) > 0$ *for all* $u \in \Omega$*, these conditions are not necessary to satisfy the Corollary 3.2.12.*

Next we provide a sufficient condition for the existence of unbounded control sets.

Theorem 3.2.15. *Consider an affine control system of the form (3.1), let* $\mathcal{C}$ *be a pathwise connected subset of the set* E_0 *of equilibria of the system (3.1) and define* $\Omega(\mathcal{C}) = \{u \in \mathrm{int}(\Omega) \mid x_u \in \mathcal{C}\}$. *Assume that*

(i) there is $u^0 \in \overline{\Omega(\mathcal{C})}$ *such that* $A(u^0)$ *has the eigenvalue* $\lambda_0 = 0$ *and* $Cu^0 + d$ *is not in the range of* $A(u^0)$*;*

(ii) every $u \in \Omega(\mathcal{C}), u \neq u^0$*, satisfies* $\mathrm{rank} A(u) = n$ *and the rank condition (3.9).*

Then, there is an unbounded control set $D \subset \mathbb{R}^n$ *containing* $\mathcal{C}$ *in the interior. More precisely, for* $u^k \in \Omega(\mathcal{C})$ *with* $u^k \to u^0$ *for* $k \to \infty$*, the equilibria* $x_{u^k} \in \mathcal{C} \subset \mathrm{int}(D)$ *satisfy for* $k \to \infty$

$$\|x_{u^k}\| \to \infty \ \textit{and} \ \frac{x_{u^k}}{\|x_{u^k}\|} \to \ker A(u^0) \cap \mathbb{S}^{n-1}. \tag{3.19}$$

Proof. Let $x_u, x_v \in \mathcal{C}$ and $h : [0,1] \longrightarrow \mathcal{C}$, with $h(0) = x_u$ and $h(1) = x_v$. Suppose that $x_v \notin \mathcal{O}^+(x_u)$ so there is $x_w \in \partial\mathcal{O}^+(x_u) \cap h([0,1])$. Thus w does not satisfy the rank condition (3.9), but this contradicts the hypothesis. So $x_v \in \mathcal{O}^+(x_u)$, by the arbitrariness of the points $\mathcal{O}^+(x_u) = \mathcal{O}^+(x_v)$, and $\mathcal{C}$ is contained in a control set D.

In order to show that D is unbounded, we argue similarly as in the scalar situation in Theorem 3.1.3.

Suppose that $u^k \in \Omega(\mathcal{C})$ converge to u^0 and assume, by way of contradiction, that x_{u^k} remains bounded. Hence we may suppose that there is $x^0 \in \mathbb{R}^n$ with $x_{u^k} \to x^0$. Then the equality

$$A(u^k)x_{u^k} = -\left[Cu^k + d\right]$$

lead for $k \to \infty$ to

$$A(u^0)x_{u^0} = -\left[Cu^0 + d\right]$$

contradicting assumption (i). We have shown that x_{u^k} becomes unbounded for $k \to \infty$. Since $Cu^k + d \to Cu^0 + d$, we get

$$A(u^k)\frac{x_{u^k}}{\|x_{u^k}\|} = \frac{1}{\|x_{u^k}\|}\left(Cu^k + d\right) \to 0.$$

On the other hand, every cluster point $y \in \mathbb{R}^n$ of the bounded sequence $\frac{x_{u^k}}{\|x_{u^k}\|}$ satisfies $\|y\| = 1$ and (3.19) follows. □

Example 3.2.16. *Consider again Example 3.1.4. In order to describe the control sets, we first check the controllability rank condition (3.9) for $|u| \neq \frac{1}{2}$. By (3.1.4)*

$$B'(u) = C + Bx_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{u}{4u^2-1}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2u \end{bmatrix} = \frac{1}{4u^2-1}\begin{bmatrix} 2u \\ -1 \end{bmatrix},$$

and hence

$$(4u^2-1)\left[B'(u), A(u)B'(u)\right] = \left[\begin{bmatrix} 2u \\ -1 \end{bmatrix}, \begin{bmatrix} 2u & 1 \\ 1 & 2u \end{bmatrix}\begin{bmatrix} 2u \\ -1 \end{bmatrix}\right] = \begin{bmatrix} 2u & 4u^2-1 \\ -1 & 0 \end{bmatrix}.$$

Thus the rank condition (3.9) holds in every equilibrium (x_u, y_u) with $|u| \neq \frac{1}{2}$.

Next we discuss the control sets for several control ranges given by a compact interval.

- Let $\Omega = [u_*, u^*]$ with $\frac{1}{2} < u_* < u^*$. Then the set of equilibria is given by the compact subset $\{(x_u, y_u) \mid u \in [u_*, u^*]\} \subset \mathcal{B}_3$. By Theorem 3.2.6 there is a single control set D_3 with $(x_u, y_u) \in \operatorname{int}(D_3)$ for all $u \in (u_*, u^*)$.

- Let $\Omega = [u_*, u^*]$ with $u_* < u^* < -\frac{1}{2}$. Then the set of equilibria is given by the compact subset $\{(x_u, y_u) \mid u \in [u_*, u^*]\} \subset \mathcal{B}_2$. By Theorem 3.2.6 there is a single closed control set D_2 with $(x_u, y_u) \in \operatorname{int}(D_2)$ for all $u \in (u_*, u^*)$.

- Let $\Omega = [u_*, u^*]$ with $-\frac{1}{2} < u_* < u^* < \frac{1}{2}$. Then the set of equilibria is given by the compact subset $\{(x_u, y_u) \mid u \in [u_*, u^*]\} \subset \mathcal{B}_1$. By Theorem 3.2.6 there is a single control set D_1 with $(x_u, y_u) \in \operatorname{int}(D_1)$ for all $u \in (u_*, u^*)$.

- Let $\Omega = [-1, 1]$. Then the connected components of the set E_0 of equilibria are

$$\mathcal{C}_1 = \left\{(x_u, y_u) \,\middle|\, u \in \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}, \quad \mathcal{C}_2 = \left\{(x_u, y_u) \,\middle|\, u \in \left(-1, -\frac{1}{2}\right)\right\},$$
$$\mathcal{C}_3 = \left\{(x_u, y_u) \,\middle|\, u \in \left(\frac{1}{2}, 1\right)\right\},$$

and there are control sets D_i with $\mathcal{C}_i \subset \operatorname{int}(D_i)$ for $i = 1, 2, 3$. Since these sets of equilibria are unbounded, the control sets are unbounded also.

We claim that they are pairwise different. For the proof, first observe that $\mathcal{B}_1$ and $\mathcal{B}_3$ are contained in $\mathcal{O}^-(x_u)$, for all $x_u \in \mathcal{B}_2$, because Lemma 3.2.9 (i) implies $\mathcal{O}^-(x_u) = \mathbb{R}^n$. Moreover, $\mathcal{B}_1$ and $\mathcal{B}_2$ are contained in $\mathcal{O}^+(x_u)$, for all $x_u \in \mathcal{B}_3$, because Lemma 3.2.9 (ii) *implies* $\mathcal{O}^+(x_u) = \mathbb{R}^n$. *It follows that one can steer the system from* D_3 *to* D_1 *and from* D_1 *to* D_2.

The control set D_2 ***is different from*** D_1 ***and*** D_3.

For the proof, we first show that for $x_{-1} = \left(-\frac{1}{3}, -\frac{2}{3}\right) \in \mathcal{B}_2$ *the reachable set satisfies* $\mathcal{O}^+(x_{-1}) \subset \left[-\frac{1}{3}, -\infty\right) \times \left[-\frac{2}{3}, -\infty\right)$. *In fact, for the right hand of the system equation in* x_{-1} *one finds for* $u \in (-1,1)$

$$2u - \frac{1}{3} + \left(-\frac{2}{3}\right) \leq -2\frac{1}{3}u - \frac{2}{3} = -\frac{2}{3}(1+u) < 0$$
$$-\frac{1}{3} + 2u\left(-\frac{2}{3}\right) + u = -\frac{1}{3} + u\left(-2\frac{2}{3}+1\right) = -\frac{1}{3} - u\frac{1}{3} = -\frac{1}{3}(1+u) < 0.$$

Consider a solution $\varphi(t, x_{-1}, u) := (\varphi_1(t, x_{-1}, u), \varphi_2(t, x_{-1}, u)), t \geq 0$, *with control values* $u(t) \in (-1,1)$ *and define*

$$\tau := \sup\left\{ t \geq 0 \,\middle|\, \varphi_1(s, x_{-1}, u) \leq -\frac{1}{3} \text{ and } \varphi_2(s, x_{-1}, u) \leq -\frac{2}{3} \text{ for all } s \in [0,t] \right\}.$$

If $\tau < \infty$, *then one finds for every* $v \in (-1,1)$

$$2v\varphi_1(\tau, x_{-1}, u) + \varphi_2(\tau, x_{-1}, u) \leq -2\frac{1}{3}v - \frac{2}{3} = -\frac{2}{3}(1+u) < 0$$
$$\varphi_1(\tau, x_{-1}, u) + 2v\varphi_2(\tau, x_{-1}, u) + v \leq -\frac{1}{3} + v\left(-2\frac{2}{3}+1\right) = -\frac{1}{3}(1+v) < 0.$$

This contradicts the definition of τ. *It follows that* $\tau = \infty$.

This shows that for any solution starting in x_{-1} *the first component of the solution cannot increase above* $-\frac{1}{3}$ *and the second component cannot increase above* $-\frac{2}{3}$ *(this holds for controls with values in* $(-1,1)$ *and then also for all controls with values in* $[-1,1]$*). Hence one cannot steer the system from* $x_{-1} \in \overline{\mathcal{B}_2'}$ *to* $\mathcal{B}_1$ *and to* $\mathcal{B}_3$. *This implies that* $D_2 \neq D_1, D_3$.

The control set D_1 ***is different from*** D_3.

For the proof, we have to show that one cannot steer the system from D_1 *to* D_3. *This follows using the phase portrait, cf. Proposition 3.1.1 and the result on asymptotics of the branches:*

For constant u the phase portrait of the affine system coincides with the phase portrait of the linear system shifting the origin to x_u; *cf. Proposition 3.1.1. Consider an initial point in* $\mathcal{B}_1$. *Since the branch* $\mathcal{B}_3$ *lies below the diagonal* $diag_1 := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_2 = z_1\}$ *and on the right of the diagonal* $diag_2 := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_2 = -z_1\}$, *it suffices to show that for every constant control u the trajectory* $\varphi(\cdot, x, u)$ *remains above* $diag_1$ *or to the right of* $diag_2$.

In Mohler [32, Fig. 1] the phase portraits are sketched for $u = -1$ *and* $u = 1$. *For*

$u \in [-1, -\frac{1}{2})$ and for $u \in (-\frac{1}{2}, 1]$ the phase portrait are analogous. For $u \in \left(-1, -\frac{1}{2}\right)$ the equilibrium $x_u \in \mathcal{B}_2$ is stable and the eigenspace for the eigenvalue $\lambda_1(u)$ is given by diag$_1$ shifted to x_u and the trajectories approach x_u tangential to this subspace. Hence the trajectories $\varphi(\cdot, x, u)$ remain above diag$_1$. For $u \in \left(\frac{1}{2}, 1\right)$ the equilibrium $x_u \in \mathcal{B}_3$ is totally unstable and the trajectories approach x_u for time tending to $-\infty$ tangential to this subspace. Here the trajectories $\varphi(\cdot, x, u)$ remain on the right of diag$_2$. For $u \in \left(-\frac{1}{2}, 0\right)$ the equilibrium $x_u \in \mathcal{B}_1 \cap \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1, z_2 > 0\}$ is hyperbolic and the unstable manifold is diag$_1$ shifted to x_u and the stable manifold is diag$_2$ shift to x_u. Hence the trajectories $\varphi(\cdot, x, u)$ remain above diag$_1$. For $u \in \left(0, \frac{1}{2}\right)$ the equilibrium $x_u \in \mathcal{B}_1 \cap \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 < 0, z_2 > 0\}$ is hyperbolic and the unstable manifold is diag$_1$ shifted to x_u and the stable manifold is diag$_2$ shifted to x_u. Hence the trajectories $\varphi(\cdot, x, u)$ remain to the right of diag$_2$. This concludes the proof that one cannot steer the system from $\mathcal{B}_1$ to $\mathcal{B}_3$ and hence also not from D_1 to D_3.

This concludes the discussion of the example.

Next we take up the linear oscillator from Example 2.2.16 and consider an associated affine control system. We will show that there are two unbounded control sets.

Example 3.2.17. *Consider the affine control system given by*

$$\ddot{x} + 3\dot{x} + (1 + u(t))x = u(t) + d \text{ with } u(t) \in [-\rho, \rho],$$

where $\rho \in \left(1, \frac{5}{4}\right)$ and $d \in \mathbb{R}$. Hence the system equation has the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + u(t) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + u(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix}.$$

For the equilibria with $u \neq -1$ we find

$$\begin{bmatrix} x_u \\ y_u \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1-u & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ u+d \end{bmatrix} = \begin{bmatrix} \frac{3}{1+u} & \frac{1}{1+u} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ u+d \end{bmatrix} = \begin{bmatrix} \frac{d+u}{1+u} \\ 0 \end{bmatrix}. \quad (3.20)$$

This yields that the connected components of the set E_0 of equilibria are

$$\mathcal{C}_1 = \left\{ \begin{bmatrix} \frac{d+u}{1+u} \\ 0 \end{bmatrix} \middle| \; u \in (-\rho, -1) \right\}, \quad \mathcal{C}_2 = \left\{ \begin{bmatrix} \frac{d+u}{1+u} \\ 0 \end{bmatrix} \middle| \; u \in (-1, \rho) \right\}.$$

For $d = 1$ there is a single equilibrium given by $(x_u, y_u)^\top = (1, 0)^\top$ for every $u \neq -1$. Henceforth we assume $d \neq 1$.

Let $d < 1$. Then for $u \in [-\rho, -1)$ one obtains $d + u < 1 + u < 0$, and for $u \in (-1, \rho]$

one obtains $1+u>0$, *hence*

$$\mathcal{C}_1=\left\{\left[\begin{array}{c}x\\0\end{array}\right]\middle|\ x\in\left(\frac{d-\rho}{1-\rho},\infty\right)\right\},\quad \mathcal{C}_2=\left\{\left[\begin{array}{c}x\\0\end{array}\right]\middle|\ x\in\left(-\infty,\frac{d+\rho}{1+\rho}\right)\right\}.$$

Let $d>1$. *Then* $u\in[-\rho,-1)$ *yields* $1+u<0$ *and* $u\in(-1,\rho]$ *yields* $1+u>0$, *hence*

$$\mathcal{C}_1=\left\{\left[\begin{array}{c}x\\0\end{array}\right]\middle|\ x\in\left(-\infty,\frac{d-\rho}{1-\rho}\right)\right\},\quad \mathcal{C}_2=\left\{\left[\begin{array}{c}x\\0\end{array}\right]\middle|\ x\in\left(\frac{d+\rho}{1+\rho},\infty\right)\right\}.$$

Note that $\mathcal{C}_1\cap\mathcal{C}_2=\varnothing$ *for all* d. *The equilibria in* $\mathcal{C}_1$ *are hyperbolic, since here* $\lambda_1(u)<0<\lambda_2(u)$ *with* $\lambda_2(u)\to 0$ *for* $u\to -1$. *The equilibria in* $\mathcal{C}_2$ *are stable nodes since here* $\lambda_1(u)<\lambda_2(u)<0$.

Next we check the assumptions of Theorem 3.2.15. For $u^0=-1$ *the matrix*

$$A(-1)=\left[\begin{array}{cc}0&1\\0&-3\end{array}\right]$$

has the eigenvalue $\lambda_0=0$ *with eigenspace* $\mathbb{R}\times\{0\}$, *and* $\operatorname{Im}A(-1)=\{(y,-3y)\mid y\in\mathbb{R}\}$. *Furthermore*

$$Cu^0+d=\left[\begin{array}{c}0\\1\end{array}\right](-1)+\left[\begin{array}{c}0\\d\end{array}\right]=\left[\begin{array}{c}0\\d-1\end{array}\right]$$

is not in the range of $A(-1)$. *This verifies assumption (i) in Theorem 3.2.15. In order to check the rank condition (3.9) we compute for* $u\neq -1$

$$B'(u)=C+B\left[\begin{array}{c}x_u\\y_u\end{array}\right]=\left[\begin{array}{c}0\\1\end{array}\right]+\left[\begin{array}{cc}0&0\\-1&0\end{array}\right]\left[\begin{array}{c}\frac{d+u}{1+u}\\0\end{array}\right]=\left[\begin{array}{c}0\\\frac{1-d}{1+u}\end{array}\right],$$
$$A(u)B'(u)=\left[\begin{array}{cc}0&1\\-1-u&-3\end{array}\right]\left[\begin{array}{c}0\\\frac{1-d}{1+u}\end{array}\right]=\left[\begin{array}{c}\frac{1-d}{1+u}\\-3\frac{1-d}{1+u}\end{array}\right].$$

Hence $\operatorname{rank}[B'(u),A(u)B'(u)]=2$ *for* $u\neq -1$. *Theorem 3.2.15 implies that there are unbounded control sets* D_i *containing the equilibria in* $\mathcal{C}_i, i=1,2$, *in the interior. For* $u^k\to u^0=-1$, *the equilibria* $(x_{u^k},y_{u^k})=(x_{u^k},0)$ *become unbounded for* $k\to\infty$ *and*

$$\frac{(x_{u^k},0)}{\|(x_{u^k},0)\|}\in\ker A(-1)\cap\mathbb{S}^1=\left\{\left[\begin{array}{c}1\\0\end{array}\right],\left[\begin{array}{c}-1\\0\end{array}\right]\right\}\text{ for all }k.$$

In the simple case considered here, the latter assertion is already clear by formula (3.20) for the equilibria.

While the asymptotic stability of the equilibria in $\mathcal{C}_2$ *implies that one can steer the system from* $\mathcal{C}_1$ *to* $\mathcal{C}_2$, *the converse does not hold, which follows by inspection of the phase portraits for the controls in* $[-\rho,-1]$ *and* $[-1,\rho]$. *It follows that* $D_1\neq D_2$.

3.3 Control sets for hyperbolic systems

In this section we present definitions of hyperbolicity and uniform hyperbolicity for affine control systems and show that hyperbolic and uniformly hyperbolic systems have a unique control set with nonvoid interior and it is bounded. Furthermore, there is a unique bounded chain control set.

Recall the definition of Floquet exponents for homogeneous periodic differential equations in (1.10) determining the exponential growth behavior of the solutions. For any τ-periodic control, the homogeneous part (3.2) of affine control system (3.1) is of this form, hence we can define corresponding Floquet exponents. They are determined by the Floquet multipliers which are the eigenvalues of the principal fundamental matrix $\Phi_u(\tau,0)$. Recall also that the system semigroup $\mathcal{S}$ of the affine control system has nonvoid interior in the system group $\mathcal{G}$.

Definition 3.3.1. *(i) For an affine control system of the form (3.1), the* ***Floquet spectrum*** *is the following set Σ_{Fl} of Floquet exponents of the homogeneous part (3.2) for controls $u \in \mathcal{U}_{pc}$ corresponding to an element $g(u) \in \operatorname{int}(\mathcal{S})$,*

$$\Sigma_{Fl} = \left\{ \lambda = \frac{1}{\tau} \log|\rho| \;\middle|\; \rho \in \operatorname{spec}(\Phi_u(\tau,0)) \cap \mathbb{R} \text{ with } g(u) \in \mathcal{S}_\tau \cap \operatorname{int}(\mathcal{S}) \right\}.$$

(ii) The affine control system (3.1) is ***hyperbolic*** *if $0 \notin \Sigma_{Fl}$. Otherwise, the system is called nonhyperbolic.*

Remark 3.3.2. *For homogeneous bilinear control system we have recalled in Definition 2.2.12 the Floquet spectrum $\Sigma_{Fl}({}_{\mathbb{P}}D)$ of control sets ${}_{\mathbb{P}}D$ with nonvoid interior in projective space $\mathbb{P}^{n-1}$. Such control sets exist, in particular, if the accessibility rank condition on $\mathbb{P}^{n-1}$ holds. We will not always need this rank condition, hence Definition 3.3.1 does not refer to control sets in $\mathbb{P}^{n-1}$. Furthermore, it will suffice to require the hyperbolicity condition only for elements g in the interior of the system semigroup $\mathcal{S}$ of the affine control system. If system (3.1) reduces to a homogeneous bilinear control system, (i.e., $C = 0$ and $d = 0$), the system semigroup $\mathcal{S}$ actually is the system semigroup ${}_{\mathbb{R}}\mathcal{S}^{\text{hom}}$ of this system on $\mathbb{R}^n \setminus \{0\}$. Under the accessibility rank condition on $\mathbb{P}^{n-1}$, Remark 2.2.13 characterizes the Floquet spectrum $\Sigma_{Fl}({}_{\mathbb{P}}D)$ using $g \in {}_{\mathbb{R}}\mathcal{S}^{\text{hom}}$ with $\pi_{\mathbb{P}}(g) \in \operatorname{int}({}_{\mathbb{P}}\mathcal{S}^{\text{hom}})$, while the Floquet spectrum from Definition 3.3.1 considers $g \in \operatorname{int}({}_{\mathbb{R}}\mathcal{S}^{\text{hom}})$.*

Remark 3.3.3. *Observe that the hyperbolicity condition $0 \notin \Sigma_{Fl}$ for the affine control system (3.1) is almost complementary to our condition for the existence of control sets with nonvoid interior in $\mathbb{R}^n \setminus \{0\}$ for its homogeneous part: This is guaranteed by Theorem 2.2.15 if $0 \in \operatorname{int}(\Sigma_{Fl}({}_{\mathbb{S}}D_i))$.*

Theorem 3.3.4. *Suppose that the system group $\mathcal{G}$ of the affine control system (3.1) acts transitively on $\mathbb{R}^n$ and that (3.1) is hyperbolic. Then there is a unique control set D with nonvoid interior. For every $g \in \operatorname{int}(\mathcal{S})$ there is a unique $x \in \mathbb{R}^n$ with $x = gx$ and*

$$D = \{x \in \mathbb{R}^n \mid \text{there is } g \in \operatorname{int}(\mathcal{S}) \text{ with } x = gx\}.$$

Proof. Let $g = g(u) \in \mathcal{S}_\tau \cap \operatorname{int}(\mathcal{S})$. By hyperbolicity, 0 is not a Floquet exponent, and hence 1 is not an eigenvalue of the principal fundamental solution $\Phi_u(\tau, 0)$. Proposition 1.2.1 (i) implies that there is a unique τ-periodic solution starting in some $x \in \mathbb{R}^n$, hence $gx = x$ by Lemma 1.4.1. By Proposition 1.3.2 (ii) it follows that $x \in \operatorname{int}(D)$ for some control set D. In order to show that D does not depend on g, consider $g, h \in \operatorname{int}(\mathcal{S})$ of the form (1.14) and the continuous path p constructed in Lemma 1.3.4 and recall Remark 1.3.5. Denote the unique fixed point of g_α by x^α. For every $\alpha \in [\sigma, 2\sigma + \tau]$, there is a control set D^α with $x^\alpha \in \operatorname{int}(D^\alpha)$, again by hyperbolicity. Since g_α and the periods τ_α depend continuously on α, Proposition 1.2.1 (iii) implies that also x^α depends continuously on α. Suppose that $x^{\alpha^*} \in \partial D$ for some $\alpha^* \in [\sigma, 2\sigma + \tau]$, as $x^{\alpha^*} \in \operatorname{int}(D^{\alpha^*})$ there is a neighborhood N of x^{α^*} such that $N \subset D^{\alpha^*}$ and $N \cap D \neq \emptyset$, this implies that $D^{\alpha^*} = D$. Therefore, $p([\sigma, 2\sigma + \tau]) \subset D$. $\square$

Example 3.3.5. *Consider the affine control system (3.16) of the Example 3.2.14. The matrices $\Phi_u(\tau, 0)$ are given by*

$$\Phi_u(\tau,0) = \begin{bmatrix} e^{\sum_{i=1}^{n}(\tau_i(a_{11}+u_i b_{11}))} & 0 \\ 0 & e^{\sum_{i=1}^{n}(\tau_i(a_{22}+u_i b_{22}))} \end{bmatrix}$$

where $u_i \in \Omega = [u_*, u^*]$, with $u_* < 0 < u^*$ and $\tau = \sum_{i=1}^{n} \tau_i$. By assumption of example we have that $a_{11} + ub_{11}, a_{22} + ub_{22} \neq 0$, for all $u \in \Omega$. Note that,

$$\Sigma_{Fl} \subset \left\{ \lambda = \frac{1}{\tau} \log |\, \rho \,| \;\middle|\; \rho \in \operatorname{spec}(\Phi_u(\tau,0)) \cap \mathbb{R} \right\},$$

so

$$\begin{aligned} \Sigma_{Fl} &\subset \left\{ \frac{1}{\tau} \log \left| e^{\sum_{i=1}^{n}(\tau_i(a_{11}+u_i b_{11}))} \right| \;\middle|\; u \in \Omega \right\} \cup \left\{ \frac{1}{\tau} \log \left| e^{\sum_{i=1}^{n}(\tau_i(a_{22}+u_i b_{22}))} \right| \;\middle|\; u \in \Omega \right\} \\ &= \left\{ \frac{1}{\tau} \sum_{i=1}^{n} (\tau_i(a_{11} + u_i b_{11})) \;\middle|\; u \in \Omega \right\} \cup \left\{ \frac{1}{\tau} \sum_{i=1}^{n} (\tau_i(a_{22} + u_i b_{22})) \;\middle|\; u \in \Omega \right\} \end{aligned}$$

since one of these holds, $a_{11} + ub_{11}) < 0$ for all $u \in \Omega$ or $a_{11} + ub_{11}) > 0$ for all $u \in \Omega$, so zero not in the first set in the union, by same arguments zero not is in the second set in the union. Thus $0 \notin \Sigma_{Fl}$ of the system (3.16), and by Theorem 3.3.4 the system has a *unique control set.*

The question arises if the control set D is bounded. We will give a positive answer provided that the following uniform hyperbolicity condition holds assuming that the control range Ω is a compact and convex neighborhood of the origin in $\mathbb{R}^m$ and hence, for system (1.6), the control flow $\mathbf{\Psi}$

$$\begin{array}{rccc} \mathbf{\Psi}: & \mathbb{R}\times\mathbb{R}^n\times\mathcal{U}_{pc} & \longrightarrow & \mathbb{R}^n\times\mathcal{U}_{pc} \\ & (t,x,u) & \longmapsto & (\varphi(t,x,u),\Theta_t u) \end{array}$$

is well defined (cf. Section 1.1).

Definition 3.3.6. *The homogeneous bilinear control system (1.6) is* ***uniformly hyperbolic*** *if the vector bundle $\mathbb{R}^n\times\mathcal{U}$ can be decomposed into the Whitney sum of two invariant subbundles $\mathcal{V}^1$ and $\mathcal{V}^2$ such that the restrictions $\mathbf{\Psi}^1$ and $\mathbf{\Psi}^2$ of the control flow $\mathbf{\Psi}$ to $\mathcal{V}^1$ and $\mathcal{V}^2$, respectively, satisfy for constants $\alpha>0$ and $K\geq 1$ and for all $(x_i,u)\in\mathcal{V}^i$*

$$\begin{aligned} \|\varphi(t,x_1,u)\| &= \left\|\mathbf{\Psi}_t^1(x_1,u)\right\| \leq Ke^{-\alpha t}\|x_1\| \ \textit{for } t\geq 0, \\ \|\varphi(t,x_2,u)\| &= \left\|\mathbf{\Psi}_t^2(x_2,u)\right\| \leq Ke^{\alpha t}\|x_2\| \ \textit{for } t\leq 0. \end{aligned}$$

Then, for $i=1,2$, one obtains that $\mathcal{V}^i(u):=\{x\in\mathbb{R}^n \mid (x,u)\in\mathcal{V}^i\}$ is a subspace of $\mathbb{R}^n$ and its dimension is independent of $u\in\mathcal{U}$. For all $u\in\mathcal{U}$

$$\mathbb{R}^n=\mathcal{V}^1(u)\oplus\mathcal{V}^2(u) \text{ and } \varphi(t,x_i,u)\in\mathcal{V}^i(u(t+\cdot)) \text{ for all } t\in\mathbb{R},$$

hence, for $x=x_1+x_2$ with $x_i\in\mathcal{V}^i(u)$ and $\Phi_u^i(t,s):=\Phi_u(t,s)_{|\mathcal{V}_i(u(s+\cdot))}$ for $t,s\in\mathbb{R}$,

$$\varphi(t,x,u)=\varphi(t,x_1,u)+\varphi(t,x_2,u) \text{ and } \Phi_u(t,s)=\Phi_u^1(t,s)+\Phi_u^2(t,s).$$

The uniform hyperbolicity condition above implies that system (3.1) is hyperbolic in the sense of Definition 3.3.1. This follows since the τ-periodic controls yield for corresponding τ-periodic trajectories $\varphi(\cdot,x,u)$ with $(x,u)\in\mathcal{V}_1$ Floquet exponents which are equal to or less than $-\alpha$, and for those with $(x,u)\in\mathcal{V}_2$ Floquet exponents which are equal to or greater than α.

Lemma 3.3.7. *Suppose that the uniform hyperbolicity assumption holds. Then there is $c>0$ such that for $(x_1,u)\in\mathcal{V}^1$ and $(x_2,u)\in\mathcal{V}^2$*

$$\|\varphi(t,x_1,u)\|\leq K\|x_1\|+\frac{Kc}{\alpha} \ \textit{for } t\geq 0, \quad \|\varphi(t,x_2,u)\|\leq K\|x_2\|+\frac{Kc}{\alpha} \ \textit{for } t\leq 0.$$

Proof. Denote the projections of $\mathbb{R}^n$ to $\mathcal{V}^1(u)$ by P_u and choose $c>0$ such that

$$\|P_u\|\,\|Cv+d\|\leq c$$

for all $u \in \mathcal{U}, v \in \Omega$. By the invariance of $\mathcal{V}^1$,

$$P_{u(t+\cdot)}\Phi_u(t,s) = \Phi_u(t,s)P_{u(s+\cdot)},$$

and hence

$$\begin{aligned}\varphi(t,x_1,u) = P_{u(t+\cdot)}\varphi(t,x_1,u) &= P_{u(t+\cdot)}\Phi_u(t,0)x_1 + \int_0^t P_{u(t+\cdot)}\Phi_u(t,s)[Cu(s)+d]ds \\ &= \Phi_u^1(t,0)x_1 + \int_0^\tau \Phi_u^1(t,s)P_{u(s+\cdot)}[Cu(s)+d]ds.\end{aligned}$$

Then it follows for all $u \in \mathcal{U}$ and $t \geq 0$ that

$$\begin{aligned}\|\varphi(t,x_1,u)\| &\leq \left\|\Phi_u^1(t,0)x_1\right\| + \int_0^t \left\|\Phi_u^1(t,s)P_{u(s+\cdot)}[Cu(s)+d]\right\| ds \\ &\leq Ke^{-\alpha t}\|x_1\| + Kc\int_0^t e^{-\alpha(t-s)}ds \leq K\|x_1\| + \frac{Kc}{\alpha}.\end{aligned}$$

The second assertion follows analogously. □

The next theorem establishes as claimed that the control set is bounded under the uniform hyperbolicity assumption.

Theorem 3.3.8. *Suppose that the accessibility rank condition holds for the affine control system (3.1) and that the homogeneous part satisfies the uniform hyperbolicity condition in Definition 3.3.6. Then the unique control set D with nonvoid interior is bounded.*

Proof. We show that $\mathrm{int}(D)$ is bounded. This will yield the assertion, since the accessibility rank condition implies that $D \subset \overline{\mathrm{int}(D)}$. Fix $x \in \mathrm{int}(D)$ and consider an arbitrary point $y \in \mathrm{int}(D) = \mathcal{O}^+(x) \cap \mathcal{O}^-(x)$. Thus there are controls $u^1, u^2 \in \mathcal{U}$ and times $t_1, t_2 > 0$ with $y = \varphi(t_1, x, u^1)$ and $y = \varphi(-t_2, x, u^2)$. Define

$$u(t) = \begin{cases} u^1(t) & \text{for } \quad t \in [0,t_1] \\ u^2(t) & \text{for } \quad t \in [-t_2, 0) \end{cases},$$

and extend u to a (t_1+t_2)-periodic function on $\mathbb{R}$. Thus $y = \varphi(t_1,x,u) = \varphi(-t_2,x,u)$ and $u(t_1+t_2+t) = u(t)$ for all $t \in \mathbb{R}$. In particular, with $t' := t-t_2$, we get $u(t_1+t') = u(-t_2+t')$ for all $t' \in \mathbb{R}$, that is, $u(t_1+\cdot) = u(-t_2+\cdot)$ in $\mathcal{U}$ implying

$$\mathcal{V}^i(u(t_1+\cdot)) = \mathcal{V}^i(u(-t_2+\cdot)) \text{ for } i = 1,2. \tag{3.21}$$

We decompose

$$x = x_1 \oplus x_2 \text{ with } x_i \in \mathcal{V}^i(u) \text{ and } y = y_1 \oplus y_2 \text{ with } y_i \in \mathcal{V}^i(u(t_1+\cdot)) \text{ for } i = 1,2.$$

The invariance of the complementary subbundles $\mathcal{V}^i$ shows that

$$y_1 = \varphi(t_1, x_1, u) = \varphi(-t_2, x_1, u) \text{ and } y_2 = \varphi(t_1, x_2, u) = \varphi(-t_2, x_2, u),$$

and by (3.21) one gets the decompositions conforming to $\mathcal{V}^1(u(t_1+\cdot)) \oplus \mathcal{V}^2(u(t_1+\cdot)) = \mathcal{V}^1(u(-t_2+\cdot)) \oplus \mathcal{V}^2(u(-t_2+\cdot))$, so

$$y = \varphi(t_1, x_1, u) \oplus \varphi(t_1, x_2, u) = \varphi(-t_2, x_1, u) \oplus \varphi(-t_2, x_2, u)$$

with

$$\begin{array}{cc} \varphi(t_1, x_1, u) \in \mathcal{V}^1(u(t_1+\cdot)), & \varphi(t_1, x_2, u) \in \mathcal{V}^2(u(t_1+\cdot)), \\ \varphi(-t_2, x_1, u) \in \mathcal{V}^1(u(-t_2+\cdot)), & \varphi(-t_2, x_2, u) \in \mathcal{V}^2(u(-t_2+\cdot)). \end{array}$$

By (3.21) this implies, in particular, that $\varphi(t_1, x_2, u) = \varphi(-t_2, x_2, u)$ and $\varphi(t_1, x_1, u) = \varphi(-t_2, x_1, u)$. Lemma 3.3.7 implies

$$\|\varphi(t_1, x_1, u)\| \leq K\|x_1\| + \frac{Kc}{\alpha} \text{ and } \|\varphi(-t_2, x_2, u)\| \leq K\|x_2\| + \frac{Kc}{\alpha}.$$

Since the bounds are independent of t_1 and t_2 these estimates hold for all $y \in \text{int}(D)$ and hence $\text{int}(D)$ is bounded. □

This result implies that the only control set of the affine control system in the Example (3.16) is bounded .

Example 3.3.9. *Consider the affine control system (3.16) of the Example 3.2.14. In the Example 3.3.5 we proved that the system have a unique control set with nonvoid interior. Now we will prove that these control set is bounded.*

There are three possibilities for $\lambda_1(u)$ and $\lambda_2(u)$:

i) $\lambda_1(u) < 0 < \lambda_2(u)$;

ii) $\lambda_1(u), \lambda_2(u) < 0$;

iii) $0 < \lambda_1(u), \lambda_2(u)$.

(i) Define invariant subbundles $\mathcal{V}^1 = (\mathbb{R} \times \{0\}) \times \mathcal{U}_{pc}$ and $\mathcal{V}^2 = (\{0\} \times \mathbb{R}) \times \mathcal{U}_{pc}$.
Let $\alpha = \min_{u \in \Omega} \{\|\lambda_1(u)\|, \|\lambda_2(u)\|\}$.
For $((x,0),u) \in \mathcal{V}^1$

$$\begin{aligned} \|\varphi(t,(x,0),u)\| &= \|\left(e^{t\lambda_1(u)}x, 0\right)\| \\ &= \sqrt{e^{2t\lambda_1(u)}x^2} \\ &= e^{t\lambda_1(u)}\|(x,0)\|, \end{aligned}$$

and for $((0,y),u) \in \mathcal{V}^2$ we have $\|\varphi(t,(0,y),u)\| = e^{t\lambda_2(u)}\|(0,y)\|$. Then, $\lambda_1(u) \leq -\alpha < \alpha < \lambda_2(u)$, so $e^{t\lambda_1(u)} \leq e^{-\alpha t}$, for all $t \geq 0$ and $e^{t\lambda_2(u)} \leq e^{\alpha t}$, for all $t \leq 0$. Furthermore, in this case the system is uniformly hyperbolic taking $K = 1$.

(ii) Take $\mathcal{V}^1 = \mathbb{R} \times \mathcal{U}_{pc}$, $\mathcal{V}^2 = \{0\} \times \mathcal{U}_{pc}$, $\alpha = \min\limits_{u\in\Omega}\{\|\lambda_1(u)\|, \|\lambda_2(u)\|\}$ and $K = 1$. Note that $\lambda_i(u) < -\alpha$ for all $u \in \Omega$. For all $((x,y),u) \in \mathcal{V}^1$ one has

$$\begin{aligned}
\|\varphi(t,(x,y),u)\| &= \sqrt{e^{2t\lambda_1(u)}x^2 + e^{2t\lambda_2(u)}y^2} \\
&\leq \sqrt{e^{-2t\alpha}x^2 + e^{-2t\alpha}y^2} \\
&= e^{-t\alpha}\|(x,y)\|
\end{aligned}$$

for all $t \geq 0$. And for $(0,u) \in \mathcal{V}^2$, one has $\|\varphi(t,(0,0),u)\| = 0$, for all $t \leq 0$. Therefore, the system is uniformly hyperbolic.

(iii) In this case the system is uniformly hyperbolic and the proof is analogous to (ii) for $\mathcal{V}^1 = \{0\} \times \mathcal{U}_{pc}$, $\mathcal{V}^2 = \mathbb{R} \times \mathcal{U}_{pc}$, $\alpha = \min\limits_{u\in\Omega}\{\|\lambda_1(u)\|, \|\lambda_2(u)\|\}$ and $K = 1$.

Now we will to determine a bounded set that contains the control set of the system (3.16).

Denote by x_M and x_m the maximum and minimum values of x_u, respectively, and y_m and y_M the maximum and minimum values of y_u, respectively (see the Example 3.2.14).

For every $(x,y) \in \mathbb{R}^2 \setminus \{[x_m, x_M] \times [y_m, y_M]\}$ there cannot be a control set containing (x,y).

Let $(x,y) \in \mathbb{R}^2 \setminus \{[x_m, x_M] \times [y_m, y_M]\}$, such that $x < x_m$. As $\det(A + uB) \neq 0$ for all $u \in \Omega$, then $\lambda_1(u)$ and $\lambda_2(u)$ are no zero. Let $t \geq 0$, one has

$$\begin{cases}
e^{t\lambda_1(u)} \leq 1 \Rightarrow e^{t\lambda_1(u)}(x - x_u) + x_u \geq x, \ \forall\ u \in \Omega, \ \text{if } \lambda_1(u) < 0 \ \forall\ u \in \Omega. \\
e^{t\lambda_1(u)} \geq 1 \Rightarrow e^{t\lambda_1(u)}(x - x_u) + x_u \leq x, \ \forall\ u \in \Omega, \ \text{if } \lambda_1(u) > 0 \ \forall\ u \in \Omega.
\end{cases}$$

Suppose that $\lambda_1(u) > 0$ for all $u \in \Omega$, and consider $(\alpha_0, \beta_0) = \varphi(t_0, (x,y), u_0)$ with $t_0 > 0$, then $\alpha_0 < x$. Let $\varepsilon = |x - \alpha_0|$. If $((\alpha_n, \beta_n))_{n\in\mathbb{N}} = (\varphi(t_n, (\alpha_0, \beta_0), u_n))_{n\in\mathbb{N}}$ is a sequence of points of $\mathcal{O}^+(\alpha_0, \beta_0)$, then $x_n \leq \alpha_0$ and $\alpha_n \notin \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right)$, for all $n \in \mathbb{N}$. Then the sequence $((\alpha_n, \beta_n))_{n\in\mathbb{N}}$ does not converge to (x,y). As the sequence is arbitrary, we conclude that $(x,y) \notin \overline{\mathcal{O}^+(\alpha_0, \beta_0)}$ to any point $(\alpha_0, \beta_0) \in \mathcal{O}^+(x,y)$. Therefore, there cannot be a controllable set that contains (x,y).

Next we present a particular example of the example above of a hyperbolic affine system.

Example 3.3.10. *Consider the system with control range $\Omega = [-1, 1]$, given by*

$$\begin{aligned}
\dot{x} &= 2x + u(t)x + 3u(t) + 3 \\
\dot{y} &= -2y + u(t)y + 3u(t).
\end{aligned} \tag{3.22}$$

Hence

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, d = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

The eigenvalue of $A(u) = A+uB$ *are given by* $\lambda_1(u) = 2+u \geq 1$ *and* $\lambda_2(u) = -2+u \leq -1$, *respectively. Thus the system is uniformly hyperbolic with* $\mathcal{V}^1 = (\{0\} \times \mathbb{R}) \times \mathcal{U}$ *and* $\mathcal{V}^2 = (\mathbb{R} \times \{0\}) \times \mathcal{U}$. *By Example 3.3.9 the unique control set with nonvoid interior is contained in* $D = [-3, 0] \times [-1, 3]$. *Indeed, the equilibria are given by*

$$0 = (2+u)x + 3u + 3, \text{ and } 0 = (-2+u)y + 3u,$$

hence

$$x_u = -\frac{3u+3}{2+u} \text{ and } y_u = \frac{3u}{2-u}.$$

The maps $u \mapsto x_u$ *and* $u \mapsto y_u$ *are monotonically decreasing and increasing, respectively, since*

$$\frac{d}{du}x_u = \frac{-3(2+u)+3u+3}{(2+u)^2} = \frac{-3}{(2+u)^2} < 0,$$

$$\frac{d}{du}y_u = \frac{3(2-u)+3u}{(2-u)^2} = \frac{6}{(2-u)^2} > 0.$$

This implies that the set of equilibria is contained in

$$[x_1, x_{-1}] \times [y_{-1}, y_1] = [-3, 0] \times [-1, 3].$$

Note that the set of equilibria is not contained in a straight line, since $(x_0, y_0) = (-\frac{3}{2}, 0)$ *is not on the line* $y = -\frac{4}{3}x - 1$ *through* $(x_{-1}, y_{-1}) = (0, -1)$ *and* $(x_1, y_1) = (-3, 3)$.

The solution for constant u and initial value $(x, y)^\top$ *has the form*

$$\varphi_1(t, x, u) = e^{\lambda_1(u)t}(x - x_u) + x_u, \quad \varphi_2(t, y, u) = e^{\lambda_2(u)t}(y - y_u) + y_u. \tag{3.23}$$

Inspection of the phase portraits for $u = -1$ *and* $u = 1$ *show that one can approximately reach (with a combination of these controls) from any point* $(x, y)^\top \in (-3, 0) \times [-1, 3]$ *any other point in this set, while this is not possible from points* $(-3, y)^\top, (0, y)^\top$ *with* $y \in [-1, 3]$. *Hence the unique control set is* $D = (-3, 0) \times [-1, 3]$.

In the figure 3.4, the red curve represent the set of equilibrium points and blue rectangle the control set of the system (3.22).

Using similar arguments, one can also show that $E = \overline{D} = [-3, 0] \times [-1, 3]$ *is the unique chain control set.*

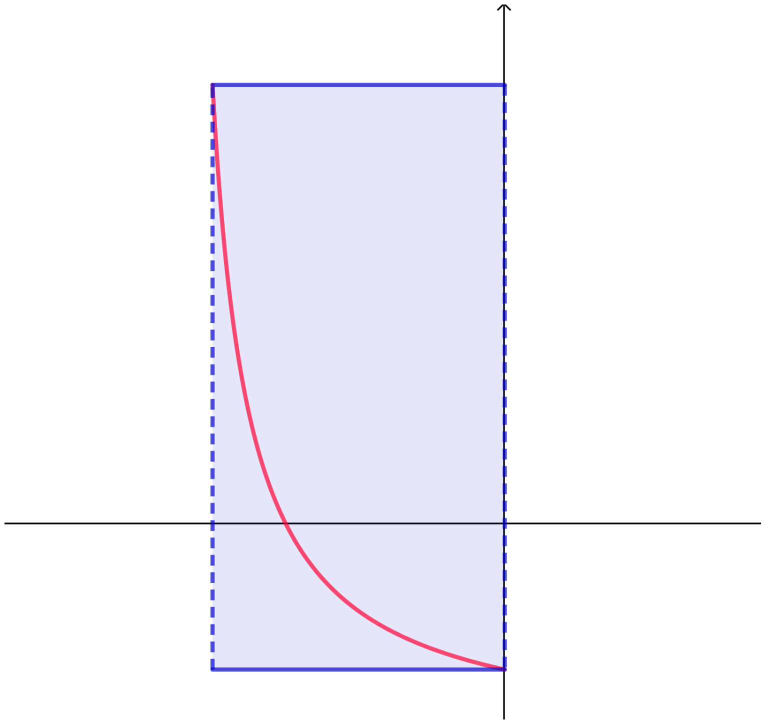

Figure 3.4: Control set of the system (3.22)

In general, for a system of the form (3.16), with $\lambda_1(u) < 0 < \lambda_2(u)$ or $\lambda_2(u) < 0 < \lambda_1(u)$, by inspecting the phase portraits, we can prove that the unique control set is $[x_m, x_M] \times (y_m, y_M)$ or $(x_m, x_M) \times [y_m, y_M]$.

Now we turn to the chain control sets. It turns out that under the uniform hyperbolicity condition there is a unique bounded chain control set and it is given by the closure of the control set with nonvoid interior. This is based on a shadowing lemma established in Colonius and Du [12].

Theorem 3.3.11. *Assume that the affine control system (3.1) satisfies the accessibility rank condition, and the homogeneous part (3.2) satisfies the uniform hyperbolicity condition in Definition 3.3.6. Then there is a unique bounded chain control set E given by the closure $\overline{D}$ of the control set D with nonvoid interior.*

Proof. **Step 1.** We show that $E := \overline{D}$ is a chain control set. Every control set with nonvoid interior is contained in a chain control set, hence there exists a chain control set E' with $D \subset E'$ and also E' has nonvoid interior. By the uniform hyperbolicity assumption and the accessibility rank condition, E' must be the closure of a control set D' by [12, Theorem 3]. Since $\mathrm{int}(D) \subset E'$ there is $x \in D' \cap D$ implying $D' = D$.

Step 2. For each $u \in \mathcal{U}$ and each bounded chain control set E', we define the u-fiber of E' by

$$E'(u) = \{x \in \mathbb{R}^n \mid \varphi(\mathbb{R}, x, u) \subset E'\} \text{ and let } \mathcal{U}(E') = \{u \in \mathcal{U} \mid E'(u) \neq \varnothing\}.$$

We prove that $\mathcal{U}(E')$ is closed in $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$ in the weak* topology. Indeed, assume that $u^k \in \mathcal{U}(E'), k \in \mathbb{N}$, and $u^k \to u \in \mathcal{U}$. Consider points $x^k \in E'(u)$. By compactness of E', we may assume that $x^k \to x \in E'$. By continuity of φ in (x, u) it follows that $\varphi(t, x, u) \in E'$ for all $t \in \mathbb{R}$ implying $E'(u) \neq \varnothing$ or, equivalently, $u \in \mathcal{U}(E')$.

Step 3. We prove that $\mathcal{U}(E) = \mathcal{U}$ for the chain control set $E = \overline{D}$. Let $u \in \mathcal{U}$ be an arbitrary control and fix $k \in \mathbb{N}$. Then there are piecewise constant $2k$-periodic controls $u^{k,i} \in \mathcal{U}_{pc}$ with $u^{k,i}_{[-k,k]} \to u_{[-k,k]}$ in the weak* topology of $L^\infty([-k, k]; \mathbb{R}^m)$. We may choose these controls such that $g(u^{k,i}(-k+\cdot)) \in \text{int}(\mathcal{S})$. Then 1 is not an eigenvalue of $\Phi_{u^{k,i}}(k, -k)$ since otherwise $0 = \frac{1}{2\tau} \log 1$ is a Floquet exponent contradicting the uniform hyperbolicity assumption. Thus

$$x(u^{k,i}) = (I - \Phi_{u^{k,i}}(k, -k))^{-1} \int_{-k}^{k} \Phi_{u^{k,i}}(k, s)[Cu^{k,i}(s) + d]ds$$

is well defined yielding a unique $2k$-periodic solution. Since $g(u^{k,i}(-k + \cdot)) \in \text{int}(\mathcal{S})$, Proposition 1.3.2(ii) shows that this solution is contained in the interior of a control set, hence, by Theorem 3.3.4, in the interior of D. In particular, the value at $t = 0$ of this solution, denoted by $y^{k,i}$, is contained in $D \subset E$. Since E is compact, we may assume that $y^{k,i}$ converges for $i \to \infty$ to an element $y^k \in E$. Thus $\varphi(t, y^{k,i}, u^{k,i}) \in E$ for $t \in [-k, k]$ and by continuity of $\varphi(t, \cdot, \cdot)$ implies that $\varphi(t, y^{k,i}, u^{k,i}) \to \varphi(t, y^k, u^k)$ for $t \in [-k, k]$. The points y^k have a cluster point $y^0 \in E$ for $k \to \infty$, and continuous dependence on the initial value implies that $\varphi(t, y^0, u) \in E$ for all $t \in \mathbb{R}$.

Step 4. Assume that E' is a bounded chain control set, different (and hence disjoint) from E. Then there are $x \in E'$ and $u \in \mathcal{U}$ with $\varphi(\mathbb{R}, x, u) \subset E'$. By Step 3 there is $y \in E$ with $\varphi(t, y, u) \in E$ for all $t \in \mathbb{R}$. Then

$$\|\varphi(t, x, u) - \varphi(t, y, u)\| = \|\Phi_u(t, 0)(x - y)\|.$$

Since $\Phi^i_{-t}(u, x_i) = (\Phi^i_t(u, x_i))^{-1}$ for $t \in \mathbb{R}$ and $i = 1, 2$, uniform hyperbolicity implies that this converges to ∞ as $t \to \infty$ or as $t \to -\infty$ contradicting the fact that both E and E' are bounded. □

3.4 Affine control systems and projective spaces

In this section we will embedded of the affine control system (3.1) to homogeneous bilinear control system on $\mathbb{R}^{n+1}$ and then projects the system on projective space $\mathbb{P}^n$, we write the projective space as the disjoint union of two subset $\mathbb{P}^{n,1}$ isomorphic to $\mathbb{R}^n$, and $\mathbb{P}^{n,0}$ isomorphic to $\mathbb{P}^{n-1}$, the projected system of homogeneous bilinear control system on $\mathbb{R}^{n+1}$ is invariant on each one these subsets and its control sets contained in $\mathbb{P}^{n,1}$ are in bijection with the control sets of the affine control system (3.1) and its control sets contained in $\mathbb{P}^{n,0}$ are in bijection with the control sets of the homogeneous part (3.2).

Systems (3.1) and (3.2) can be embedded into a homogeneous bilinear control system in $\mathbb{R}^{n+1}$ of the form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A & d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \sum_{i=1}^{m} u_i(t) \begin{bmatrix} B_i & c_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{3.24}$$

Denote the solutions of (3.24) with initial condition $(x(0), z(0)) = (x^0, z^0) \in \mathbb{R}^n \times \mathbb{R}$ by $\psi(t, (x^0, z^0), u)$, $t \in \mathbb{R}$. For initial condition of the form $(x^0, 0)$ or $(x^0, 1)$, the solution $\psi(t, (x^0, z^0), u)$ is related to solution of the system (3.1) or of the system (3.2). Denote the solution of the system (3.1) with initial condition x_0 and control u by $\varphi(t, x_0, u)$ and the solution of the system (3.2) by $\varphi_{hom}(t, x^0, u)$.

For initial values of the form $(x^0, 1) \in \mathbb{R}^{n+1}$ one finds for (3.24)

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) \left(B_i x(t) + c_i\right) + d, \quad \dot{z}(t) = 0,$$

and hence

$$\psi\left(t, \left(x^0, 1\right), u\right) = \left(\varphi(t, x^0, u), 1\right) \in \mathbb{R}^{n+1}. \tag{3.25}$$

For initial values of the form $(x^0, 0) \in \mathbb{R}^{n+1}$ one finds

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) B_i x(t), \quad \dot{z}(t) = 0,$$

and hence

$$\psi\left(t, \left(x^0, 0\right), u\right) = \left(\varphi_{\text{hom}}\left(t, x^0, u\right), 0\right) \in \mathbb{R}^{n+1}. \tag{3.26}$$

Thus the trajectories (3.25) and (3.26) are copies of the trajectories of (3.1) and of its homogeneous part (3.2), respectively, obtained by adding a trivial $(n+1)$ st. component. Note that $\alpha\psi(t, (x^0, 1), u) = (\alpha\varphi(t, x^0, u), \alpha)$ for $\alpha \in \mathbb{R}$. In general, $\alpha\varphi(t, x^0, u) \neq \varphi(t, \alpha x^0, u)$, and for initial values of the form $(x^0, z^0) \in \mathbb{R}^{n+1}$ with $z^0 \neq 0, 1$ the solutions of (3.24) are not related to those of (3.1) or (3.2).

In order to distinguish explicitly between control sets and chain control sets referring to the affine control system and its homogeneous part, we will mark the latter by the suffix "hom" in this section.

An immediate consequence of the formulas above is the following proposition.

Proposition 3.4.1. *(i) A subset $D \subset \mathbb{R}^n$ is a control set of (3.1) if and only if the set $D^1 := \{(x, 1) \mid x \in D\}$ is a control set of (3.24) in $\mathbb{R}^{n+1} \setminus \{0\}$.*

(ii) A subset ${}_{\mathbb{R}}D^{\text{hom}} \subset \mathbb{R}^n \setminus \{0\}$ is a control set of (3.2) if and only if the set $D^0 := \{(x, 0) \mid x \in {}_{\mathbb{R}}D^{\text{hom}}\}$ is a control set of (3.24) in $\mathbb{R}^{n+1} \setminus \{0\}$.

Observe that the control sets D^1 and D^0 of system (3.24) considered in Proposition 3.4.1 are contained in the invariant affine hyperplanes $\mathbb{R}^n \times \{1\}$ and $\mathbb{R}^n \times \{0\}$, respectively. In particular, all control sets D^1 and D^0 have void interiors.

Next we discuss associated systems in projective spaces. Recall that $\mathbb{P}^{n-1} = (\mathbb{R}^n \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation $x \sim y$ if $y = \lambda x$ with some $\lambda \neq 0$. An atlas of $\mathbb{P}^{n-1}$ is given by n charts (U_i, ψ_i), where U_i is the set of equivalence classes $[x_1 : \cdots : x_n]$ with $x_i \neq 0$ (using homogeneous coordinates) and $\psi_i : U_i \to \mathbb{R}^{n-1}$ is defined by

$$\psi_i([x_1 : \cdots : x_n]) = \left(\frac{x_1}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right);$$

here the hat means that the i-th entry is missing. Denote by $\pi_{\mathbb{P}}$ both projections $\mathbb{R}^n \to \mathbb{P}^{n-1}$ and $\mathbb{R}^{n+1} \to \mathbb{P}^n$.

A metric on $\mathbb{P}^n$ is given by defining for elements $p_1 = \pi_{\mathbb{P}}(x)$, $p_2 = \pi_{\mathbb{P}}(y)$

$$d(p_1, p_2) = \min\left\{\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|, \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right\}. \tag{3.27}$$

Homogeneous bilinear control systems induce control systems on projective space which we also call the projected system. In particular, one can project system (3.24) in $\mathbb{R}^{n+1}$ to projective space $\mathbb{P}^n$. In homogeneous coordinates it is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \left(\begin{bmatrix} A & d \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^{m} u_i(t) \begin{bmatrix} B_i & c_i \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{3.28}$$

Projective space $\mathbb{P}^n$ can be written as the disjoint union $\mathbb{P}^n = \mathbb{P}^{n,1} \dot{\cup} \mathbb{P}^{n,0}$, where, in homogeneous coordinates, the levels $\mathbb{P}^{n,i}$ are given by

$$\mathbb{P}^{n,i} := \{[x_1 : \cdots : x_n : i] \mid (x_1, \ldots, x_n) \in \mathbb{R}^n\} \quad \text{for} \quad i = 0, 1.$$

Observe that, by homogeneity,

$$\mathbb{P}^{n,0} = \{[x_1 : \cdots : x_n : 0] \mid \ \|(x_1, \ldots, x_n)\| = 1\}.$$

Any trajectory of system (3.28) is obtained as the projection of a trajectory of (3.24) with initial condition satisfying $z^0 = 0$ or 1, since any initial value $[x_1^0 : \cdots : x_n^0 : z^0]$ with $z^0 \neq 0$ coincides with $\left[\frac{x_1^0}{z^0} : \cdots : \frac{x_n^0}{z^0} : 1\right]$.

Loosely speaking, $\mathbb{P}^{n,0}$ is projective space $\mathbb{P}^{n-1}$(embedded into $\mathbb{P}^n$) and $\mathbb{P}^{n,1}$ is $\mathbb{P}^n$ without $\mathbb{P}^{n-1}$. The following observations make this more precise. As noted above, an atlas of $\mathbb{P}^n$ is given by $n+1$ charts (U_i, ψ_i). A trivial atlas for $\mathbb{P}^{n,1}$ is given by $\{(U_{n+1}, \psi_{n+1})\}$ proving that $\mathbb{P}^{n,1}$ is a manifold which is diffeomorphic to $\mathbb{R}^n$. The space $\mathbb{P}^{n,0}$ is closed in

$\mathbb{P}^n$, and the spaces $\mathbb{P}^{n-1}$ and $\mathbb{P}^{n,0}$ are diffeomorphic under the map

$$e: \begin{array}{ccc} \mathbb{P}^{n-1} & \longrightarrow & \mathbb{P}^{n,0} \\ [x_1 : \cdots : x_n] & \longmapsto & [x_1 : \cdots : x_n : 0] \end{array}. \tag{3.29}$$

Given $x \in \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n = T_x\mathbb{R}^n$ there is a smooth curve $\alpha(t)$ in $\mathbb{R}^n$ such that $\alpha'(0) = v$ and $\alpha(0) = x$. Hence $\psi_{n+1}^{-1}(\alpha(t)) = [\alpha(t) : 1]$ is a smooth curve in $\mathbb{P}^{n,1}$ and $\left(\psi_{n+1}^{-1} \circ \alpha\right)'(t) = [\alpha'(t) : 0]$.

Consider, for $u \in \Omega$, the affine vector field on $\mathbb{R}^n$,

$$X_u(x) = Ax + d + \sum_{i=1}^m u_i \left(B_i x + c_i\right).$$

The vector field in $\mathbb{P}^{n,1}$ which is ψ_{n+1}^{-1}-related with X_u is given by

$$\left(d\psi_{n+1}^{-1}\right)_x X_u(x) = X_u(x).$$

For any trajectory $\psi(t, (x^0, 1), u) = (\varphi_1(t, x^0, u), \ldots, \varphi_n(t, x^0, u), 1)$ of system (3.24) in $\mathbb{R}^{n+1} \setminus \{0\}$; cf. formula (3.25), the projection to $\mathbb{P}^{n,1} \subset \mathbb{P}^n$ has homogeneous coordinates

$$\left[\varphi_1\left(t, x^0, u\right) : \cdots : \varphi_n\left(t, x^0, u\right) : 1\right]. \tag{3.30}$$

The ensuing proposition discusses the control sets and chain control sets in projective spaces.

Proposition 3.4.2. *Consider in $\mathbb{R}^n$ the affine control system (3.1), its homogeneous part (3.2), and in $\mathbb{R}^{n+1}$ the homogeneous bilinear control system (3.24) as well as the projected system in $\mathbb{P}^{n-1}$ induced by (3.2) and the system (3.28) in $\mathbb{P}^n$ induced by (3.24).*

(i) Every control set $D \subset \mathbb{R}^n$ of the affine control system (3.1) yields a control set ${}_{\mathbb{P}}D^1 = \pi_{\mathbb{P}}\left(D^1\right)$ of the system (3.28) in $\mathbb{P}^n$ via the map

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{P}^{n,1} \subset \mathbb{P}^n \\ (x_1, \ldots, x_n) & \longmapsto & [x_1 : \cdots : x_n : 1]. \end{array}$$

Furthermore, D is an invariant control set if and only if ${}_{\mathbb{P}}D^1$ is an invariant control set. The control set D is unbounded if and only if the boundary of ${}_{\mathbb{P}}D^1$ satisfies $\partial\left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0} \neq \varnothing$. More precisely, if $x^k \in D$ with $\left\|x^k\right\| \to \infty$, then every cluster point y of $\frac{x^k}{\left\|x^k\right\|}$ satisfies, in homogeneous coordinates,

$$\left[x_1^{k_i} : \cdots : x_n^{k_i} : 1\right] \to [y_1 : \cdots : y_n : 0] \quad \text{for a subsequence } k_i \to \infty.$$

(ii) Every control set ${}_{\mathbb{P}}D^{\hom} \subset \mathbb{P}^{n-1}$ of the system in $\mathbb{P}^{n-1}$ induced by (3.2) corresponds to

a unique control set $e({}_{\mathbb{P}}D^{\text{hom}})$ of the system (3.28) restricted to $\mathbb{P}^{n,0}$ and conversely, via the map (3.29). There are no further control sets in $\mathbb{P}^{n,0}$.

(iii) *Every chain control set ${}_{\mathbb{P}}E^{\text{hom}}$ of the system in $\mathbb{P}^{n-1}$ induced by (3.2) corresponds to a unique chain control set ${}_{\mathbb{P}}E^{0} = e\left({}_{\mathbb{P}}E^{\text{hom}}\right)$ of the system (3.28) restricted to $\mathbb{P}^{n,0}$ and conversely, via the map (3.29). There are no further chain control sets in $\mathbb{P}^{n,0}$.*

Proof. (i) The formula (3.30) implies that ${}_{\mathbb{P}}D^1$ is a control set of the system (3.28). For a control set $D \subset \mathbb{R}^n$ of (3.1), Proposition 3.4.1 (i) implies that the projection to $\mathbb{P}^n$ of the corresponding control set D^1 in $\mathbb{R}^{n+1} \setminus \{0\}$ is given in homogeneous coordinates by

$$ {}_{\mathbb{P}}D^1 = \{[x_1 : \cdots : x_n : 1] \mid (x_1, \ldots, x_n) \in D\} \subset \mathbb{P}^{n,1}. $$

If $x^k \in D$ with $\|x^k\| \to \infty$, then ${}_{\mathbb{P}}D^1$ will contain the sequence with homogeneous coordinates $[x_1^k : \cdots : x_n^k : 1]$ satisfying $\|x^{k_i}\| \to \infty$ for some subsequence $k_i \to \infty$. Since $\frac{1}{\|x^{k_i}\|} \to 0$ for $k_i \to \infty$, this sequence satisfies

$$ [x_1^{k_i} : \cdots : x_n^{k_i} : 1] = \left[\frac{x_1^{k_i}}{\|x^{k_i}\|} : \cdots : \frac{x_n^{k_i}}{\|x^{k_i}\|} : \frac{1}{\|x^{k_i}\|}\right] \to [y_1 : \cdots : y_n : 0] \quad \text{for} \quad k_i \to \infty. $$

In particular, the boundary of ${}_{\mathbb{P}}D^1$ in $\mathbb{P}^n$ satisfies $\partial\left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0} \neq \varnothing$.

(ii) Consider in homogeneous coordinates an element $[x_1 : \cdots : x_n] \in {}_{\mathbb{P}}D^{\text{hom}}$. In local coordinates $(U_i, \psi_i), i \in \{1, \ldots, n\}$, this element is

$$ \psi_i([x_1 : \cdots : x_n]) = \left(\frac{x_1}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right) \text{ if } x_i \neq 0. $$

Then, for any $u \in \mathcal{U}$, the corresponding solution $\varphi_{\text{hom}}(t, x, u)$ of the homogeneous equation (3.2) in $\mathbb{R}^n$ coincides with the first n components of the solution $\psi(t, (x, 0), u)$ of (3.24) with initial state $(x, 0)$. For $x \neq 0$, we project this solution to $\mathbb{P}^n$ and get in homogeneous coordinates

$$ [\varphi_{\text{hom},1}(t, x, u) : \cdots : \varphi_{\text{hom},n}(t, x, u) : 0] \in \mathbb{P}^{n,0}. $$

It follows that any control set ${}_{\mathbb{P}}D^{\text{hom}}$ of the system induced in projective space $\mathbb{P}^{n-1}$ by the homogeneous part (3.2) of (3.1) yields a control set ${}_{\mathbb{P}}D^0$ contained in $\mathbb{P}^{n,0}$ of the system induced in projective space $\mathbb{P}^n$ by (3.24). We can also directly embed the control set ${}_{\mathbb{P}}D \subset \mathbb{P}^{n-1}$ into $\mathbb{P}^{n,0}$ and obtain the control set ${}_{\mathbb{P}}D^0 = e({}_{\mathbb{P}}D)$.

Assertion (iii) follows as (ii) taking into account the metric (3.27) on $\mathbb{P}^{n,0}$. □

We remark that the assertion in Proposition 3.4.2 (ii) also holds, if the accessibility rank condition in $\mathbb{P}^{n-1}$ is not valid (this is the case in Example 3.5.8).

Example 3.4.3. *Consider the system (3.2.8), which* $0 < \rho \in \mathbb{R}$ *and* $\Omega = [-\rho, \rho]$ *given by*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} + u \begin{pmatrix} c \\ d \end{pmatrix} \tag{3.31}$$

with $a, b, c, d > 0$.

We describe their equilibrium points (3.2.8), given by two components $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *and the control sets* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *that contain them. If we consider the homogeneous system in* $\mathbb{R}^{n+1}$ *we have that* $D_1^1 = \mathcal{C}_1 \times \{1\}$ *and* $D_2^1 = \mathcal{C}_2 \times \{1\}$ *are control sets contains the equilibria* $\mathcal{C}_1 \times \{1\}$ *and* $\mathcal{C}_2 \times \{1\}$*, respectively. Then we projects this control sets in* $\mathbb{P}^n$ *and obtains two control set given by* $\pi_{\mathbb{P}}(D_1^1)$ *and* $\pi_{\mathbb{P}}(D_2^1)$,

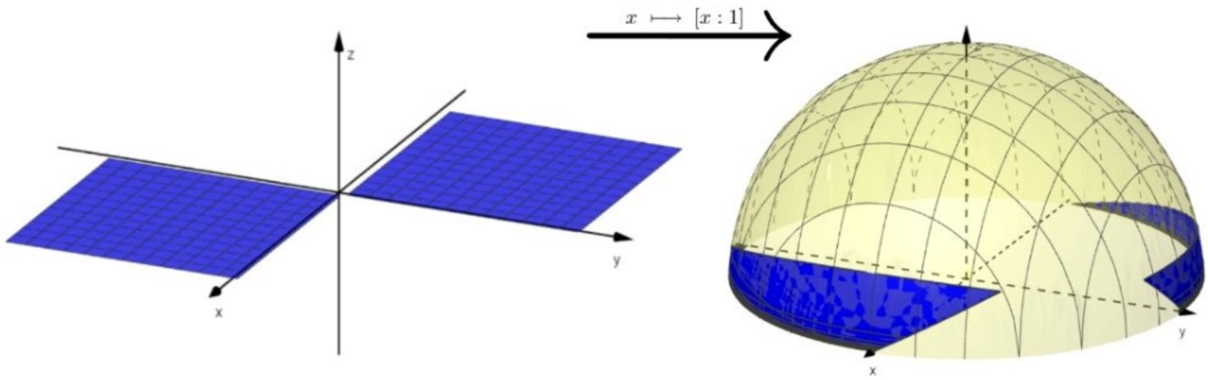

Figure 3.5: Equilibria in $\mathbb{R}^n$ and in $\mathbb{P}^n$

Furthermore, is we consider its homogeneous part projected in the projective space $\mathbb{P}^1$ *we have that the system has four control sets:* $\{[0:1]\}, \{[1:0]\}, \{[x]; 0 < x < 1\}$ *and* $\{[x]; -1 < x < 0\}$.

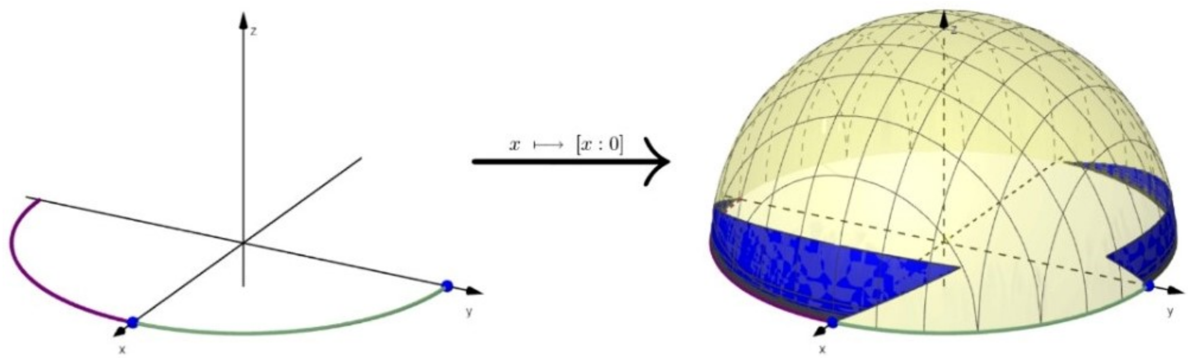

Figure 3.6: Equilibria in $\mathbb{P}^{n-1}$ and in $\mathbb{P}^n$

The intersection $\partial\left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0}$ will be of relevance below. Hence we give it a suggestive name.

Definition 3.4.4. *For a control set $D \subset \mathbb{R}^n$ with associated control set ${}_{\mathbb{P}}D^1$ in $\mathbb{P}^{n,1}$ the set $\partial_\infty(D) := \partial\left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0}$ is the boundary at infinity of D.*

Proposition 3.4.2 (i) shows, in particular, that the boundary at infinity $\partial_\infty(D)$ of an unbounded control set D is nonvoid.

Remark 3.4.5. *The construction of the boundary at infinity in $\mathbb{P}^{n,0}$ of a control set in $\mathbb{P}^{n,1}$ has some similarity to the ideal boundary used by Firer and Do Rocio [23] in the analysis of invariant control sets for sub-semigroups of a semisimple Lie group. For ordinary differential equations, the study of the behavior at infinity is a classical topic based on the Poincaré sphere; cf. Perko [34, Section 3.10].*

For hyperbolic systems one easily obtains the following result on the control sets in $\mathbb{P}^{n,1}$.

Corollary 3.4.6. *Suppose that the accessibility rank condition holds for the affine control system (3.1). If (3.1) is hyperbolic the unique control set $D \subset \mathbb{R}^n$ yields a control set ${}_{\mathbb{P}}D^1$ in $\mathbb{P}^{n,1}$. If the uniform hyperbolicity property in Definition 3.3.6 holds, the control set D is bounded in $\mathbb{R}^n$ and the closure of ${}_{\mathbb{P}}D^1$ taken in $\mathbb{P}^n$ satisfies $\overline{\pi_{\mathbb{P}}D^1} \cap \mathbb{P}^{n,0} = \varnothing$.*

Proof. This is a consequence of Theorem 3.3.4, Theorem 3.3.8, and Proposition 3.4.2. □

Next we clarify the relations between the accessibility rank conditions on the relevant spaces. This is based on the following results in Bacciotti and Vivalda [3, Section 4]. Consider a matrix $\widetilde{A} \in \mathbb{R}^{d\times d}$ and let $z = (z_1, \ldots, \widehat{z_i}, \ldots, z_d) \in \psi_i(U_i)$. Then the projection of the linear vector field $\widetilde{A}x$ to $\mathbb{P}^{d-1}$ is given in local coordinates on $\psi_i(U_i)$ by

$$\widetilde{A}^i z = (a_1(\bar{z}^i), \ldots, \widehat{a_i(\bar{z}^i)}, \ldots, a_d(\bar{z}^i))^\top - a_i(\bar{z}^i)z, \tag{3.32}$$

where

$$\bar{z}^i = (z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_d) \tag{3.33}$$

and $a_1(x), \ldots, a_d(x)$ are the d components of $\widetilde{A}x$, i.e., $\widetilde{A}x = (a_1(x), \ldots, a_d(x))$. Furthermore, the Lie bracket of the projections of two linear vector fields $\widetilde{A}$ and $\widetilde{B}$ is equal to the projection of the Lie bracket of two linear vector fields $\widetilde{A}$ and $\widetilde{B}$. For a family $\mathcal{F}_{\text{lin}}$ of linear vector fields on $\mathbb{R}^d$, the Lie algebra generated by the projections to $\mathbb{P}^{d-1}$ of the vector fields in $\mathcal{F}_{\text{lin}}$ is given by the projection on $\mathbb{P}^{d-1}$ of the Lie algebra generated by $\mathcal{F}_{\text{lin}}$.

We will apply this to matrices in $\mathbb{R}^{(n+1)\times(n+1)}$ with $u \in \Omega$ of the form

$$\widetilde{A} = \begin{bmatrix} A & d \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^m u_i \begin{bmatrix} B_i & c_i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A + \sum_{i=1}^m u_i B_i & d + \sum_{i=1}^m u_i c_i \\ 0 & 0 \end{bmatrix},$$

and we evaluate the vector field on $\mathbb{P}^n$ in a point with homogeneous coordinates

$$[x_1 : \cdots : x_n : 1] \in U_{n+1} = \mathbb{P}^{n,1}.$$

For the last component we find that $a_{n+1}(x,z) = 0$, since

$$\widetilde{A}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \left(A + \sum_{i=1}^m u_i B_i\right)x + \left(d + \sum_{i=1}^m u_i c_i\right)z \\ 0 \end{bmatrix}.$$

Furthermore, for $i = n+1$, the vector in (3.33) has the form $(x,1)$ and the vector in (3.32) is given by

$$\begin{aligned} \widetilde{A}^{n+1}\begin{bmatrix} x \\ z \end{bmatrix} &= \begin{bmatrix} A + \sum\limits_{i=1}^m u_i B_i & d + \sum\limits_{i=1}^m u_i c_i \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} \\ &= \left[A + \sum_{i=1}^m u_i B_i\right]x + \left[d + \sum_{i=1}^m u_i c_i\right]. \end{aligned}$$

We obtain the following result on the accessibility rank conditions.

Theorem 3.4.7. *(i) If the accessibility rank condition holds for affine control system (3.1) on $\mathbb{R}^n$, then the accessibility rank condition also holds for the system on the submanifold $\mathbb{P}^{n,1} \subset \mathbb{P}^n$ induced by the bilinear system (3.24) on $\mathbb{R}^{n+1}$.*

(ii) If the accessibility rank condition holds for the system on $\mathbb{P}^{n-1}$ induced by the homogeneous part of system (3.1), then it holds for the system on the invariant submanifold $\mathbb{P}^{n,0} \subset \mathbb{P}^n$ induced by the bilinear system (3.24) on $\mathbb{R}^{n+1}$.

(iii) If the assumptions in (i) and (ii) hold, then the system on $\mathbb{P}^n$ induced by the bilinear system (3.24) on $\mathbb{R}^{n+1}$ has two maximal integral manifolds: The manifold $\mathbb{P}^{n,1}$ with dimension n and the manifold $\mathbb{P}^{n,0}$ with dimension $n-1$. On both maximal integral manifolds, the respective accessibility rank condition is satisfied.

Proof. First we compute for the linear vector fields determining control system (3.24) on $\mathbb{R}^{n+1}$ the corresponding vector fields on $\mathbb{P}^n$ in local coordinates. Consider for $u \in \Omega$ the linear vector field on $\mathbb{R}^{n+1}$ given by

$$\breve{A} = \begin{bmatrix} A & d \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^m u_i \begin{bmatrix} B_i & c_i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A + \sum_{i=1}^m u_i B_i & d + \sum_{i=1}^m u_i c_i \\ 0 & 0 \end{bmatrix}, \tag{3.34}$$

and evaluate the induced vector field on $\mathbb{P}^n$ in a point with homogeneous coordinates $[x_1 : \cdots : x_n : 1] \in U_{n+1} = \mathbb{P}^{n,1}$. For the last component we find that $a_{n+1}(x,z) = 0$, since

$$\breve{A}(x,z)^\top = \left(\left(A + \sum_{i=1}^m u_i B_i\right)x + \left(d + \sum_{i=1}^m u_i c_i\right)z, 0\right)^\top.$$

Furthermore, for $i = n+1$, the vector in (3.32) is given by

$$\breve{A}^{n+1}(x,1)^{\top} = \left(A + \sum_{i=1}^{m} u_i B_i\right) x + \left(d + \sum_{i=1}^{m} u_i c_i\right).$$

Then assertion (i) follows since the vector fields for affine system (3.1) on $\mathbb{R}^n$ and the local expressions for the vector fields on $\mathbb{P}^{n,1}$ coincide.

(ii) We will compare the local coordinates for the vector fields on $\mathbb{P}^{n-1}$ and on $\mathbb{P}^{n,0} \subset \mathbb{P}^n$. Consider a linear vector field on $\mathbb{R}^n$ corresponding to the homogeneous part of system (3.1),

$$\widetilde{A} = A + \sum_{i=1}^{m} u_i B_i, \quad u \in \Omega.$$

As described above, for every $x \in \mathbb{R}^n$ with $x_i \neq 0$ the projection of the linear vector field $\widetilde{A}x = (a_1(x), \ldots, a_n(x))$ to a vector field on $\mathbb{P}^{n-1}$ is given in local coordinates on $\psi_i(U_i)$ by

$$\widetilde{A}^i z = (a_1(\bar{z}^i), \ldots, \widehat{a_i(\bar{z}^i)}, \ldots a_n(\bar{z}^i))^{\top} - a_i(\bar{z}^i) z,$$

where $\bar{z}^i = (z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{\top}$. Next, consider a vector field (3.34) for the bilinear system (3.24) on $\mathbb{R}^{n+1}$. For the charts of $\mathbb{P}^n$ restricted to $\mathbb{P}^{n,0} = e(\mathbb{P}^{n-1})$ observe

$$\breve{\psi}_i(\breve{z}) = (\psi_i(x), 0)^{\top} \quad \text{for } \breve{z} = [x_1 : \cdots : x_n : 0] \in \breve{U}_i \cap \mathbb{P}^{n,0} = U_i \times \{0\} \text{ with } x_i \neq 0.$$

For these vectors $\breve{z} \in \mathbb{P}^{n,0}$ the projection of the linear vector field $\breve{A}\breve{z}$ to a vector field on $\mathbb{P}^n$ is given in local coordinates on $\breve{\psi}_i(\breve{U}_i \cap \mathbb{P}^{n,0})$ by

$$\breve{A}^i \breve{z} = (\breve{a}_1(\tilde{z}^i), \ldots, \widehat{\breve{a}_i(\tilde{z}^i)}, \ldots, \breve{a}_n(\tilde{z}^i), \breve{a}_{n+1}(\tilde{z}^i))^{\top} - \breve{a}_i(\tilde{z}^i)\breve{z},$$

where $\breve{z} = (\breve{z}_1, \ldots, \breve{z}_n, 0)^{\top}$, $\tilde{z}^i = (\breve{z}_1, \ldots, \breve{z}_{i-1}, 1, \breve{z}_{i+1}, \ldots, \breve{z}_n, 0)^{\top}$, and

$$\breve{A}\breve{z} = (\breve{a}_1(\breve{z}), \ldots, \breve{a}_n(\breve{z}), \breve{a}_{n+1}(\breve{z}))^{\top} = (a_1(z), \ldots, a_n(z), 0)^{\top}.$$

Thus we get

$$\breve{A}^i z = (a_1(\bar{z}^i), \ldots, \widehat{a_i(\bar{z}^i)}, \ldots, a_n(\bar{z}^i), 0) - a_i(\bar{z}^i)(z_1, \ldots, z_n, 0) = \left(\widetilde{A}^i z, 0\right).$$

Hence the local coordinates of $\breve{A}^i$ and of $\widetilde{A}^i$ coincide and assertion (ii) follows.

(iii) The rank conditions in (i) and (ii) imply that the system group acts transitively on $\mathbb{P}^{n,1}$ and on $\mathbb{P}^{n,0}$. □

3.5 Control sets for nonhyperbolic systems

This section shows that all control sets with nonvoid interior are unbounded if the hyperbolicity condition specified in Definition 3.3.1 is not valid. Using the compactification of the state space constructed in the previous section, we analyze in detail when the boundary at infinity of a control set with nonvoid interior of the affine control system in $\mathbb{P}^n$ intersects the image of a control set or a chain control set of the homogeneous part. Finally, it is shown that there is a single chain control set in $\mathbb{P}^n$ containing the images of all control sets D with nonvoid interior in $\mathbb{R}^n$, and the boundary at infinity of this chain control set contains all chain control sets of the homogeneous part having nonvoid intersection with the boundary at infinity of one of the control sets D.

We start with the following motivation. Consider a linear control system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) \in \Omega, \tag{3.35}$$

where the control range $\Omega \subset \mathbb{R}^m$ is a compact convex neighborhood of the origin. This is a special case of system (1.1) for $B_1 = \cdots = B_m = 0$ and $d = 0$. We assume that the system without control restriction is controllable. By Colonius and Kliemann [13, Example 3.2.16] there is a unique control set D with nonvoid interior. With $\mathbf{GE}(A;\mu)$ denoting the real generalized eigenspace for an eigenvalue μ of A it satisfies

$$\mathbf{E}_0 \subset D \subset K + \mathbf{E}_0 + F, \tag{3.36}$$

where $\mathbf{E}_0 := \bigoplus_{\operatorname{Re}\mu=0} \mathbf{GE}(A;\mu)$ is the central spectral subspace, $K \subset \bigoplus_{\operatorname{Re}\mu<0} \mathbf{GE}(A;\mu)$ and $F \subset \bigoplus_{\operatorname{Re}\mu>0} \mathbf{GE}(A;\mu)$, the sets K and F are compact (cf. Hinrichsen and Pritchard [26, Theorem 6.2.23]; this can also be deduced from Sontag [39, Corollary 3.6.7]). Thus D is bounded if and only if $\mathbf{E}_0 = \{0\}$, i.e., if A is hyperbolic. If A is nonhyperbolic we embed system (3.35) into a homogeneous bilinear control system in $\mathbb{R}^{n+1}$ as explained in Section 3.4 and, for the corresponding control set ${}_{\mathbb{P}}D^1$ in $\mathbb{P}^n$, we find that the boundary at infinity satisfies

$$\partial_\infty(D) = \partial\left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0} = \{[x_1 : \cdots : x_n : 0] \mid [x_1 : \cdots : x_n] \in \pi_{\mathbb{P}}\mathbf{E}_0\}. \tag{3.37}$$

This follows from (3.36) noting that for $0 \neq x \in \mathbf{E}_0$ and every $j \in \mathbb{N}$ one obtains an element of D given by

$$k_j + jx + f_j \text{ with } k_j \in K, f_j \in F \text{ and } x \in \mathbf{E}_0.$$

Considering the homogeneous coordinates and dividing by j one finds for $j \to \infty$ that (3.37) holds. The set $\pi_{\mathbb{P}}\mathbf{E}_0$ is a maximal invariant chain transitive set for the flow induced by the homogeneous part $\dot{x} = Ax$ on $\mathbb{P}^{n-1}$ (cf. Colonius and Kliemann [14, Theorem 4.1.3]). Thus the boundary at infinity $\partial_\infty(D)$ is a maximal invariant chain transitive set for the induced flow on $\mathbb{P}^{n,0}$.

For general affine control systems of the form (3.1) it stands to reason to replace the maximal chain transitive sets $\pi_{\mathbb{P}}\mathbf{E}_0$ by maximal chain transitive sets of the control flow associated with the homogeneous part or, equivalently, by chain control sets in $\mathbb{P}^{n-1}$ (cf. (2.19)) and to replace the spectral property of $\mathbf{E}_0$ by appropriate generalized spectral properties. However, the situation for affine control systems will turn out to be more intricate than for linear control systems.

Now we start our discussion of the nonhyperbolic case. Here several control sets with nonvoid interior may coexist as illustrated by Examples 3.2.16 and 3.2.17 . The following theorem shows that in the nonhyperbolic case all control sets with nonvoid interior are unbounded. Observe that 0 is a Floquet exponent for the differential equation in (3.2) with τ-periodic control u if and only if $1 \in \mathrm{spec}(\Phi_u(\tau,0))$.

Theorem 3.5.1. *Assume that the affine control system (3.1) on $\mathbb{R}^n$ satisfies the accessibility rank condition. If the system is nonhyperbolic every control set D with nonvoid interior is unbounded.*

More precisely, for every control set D with nonvoid interior there are $g(u) \in \mathcal{S}_\tau \cap \mathrm{int}(\mathcal{S}), \tau > 0$, with $1 \in \mathrm{spec}(\Phi_u(\tau,0))$ and points $x^k \in \mathrm{int}(D)$ such that

$$\left\|x^k\right\| \to \infty \text{ and } d\left(\frac{x^k}{\|x^k\|}, \mathbf{E}(\Phi_u(\tau,0);1)\right) \to 0 \text{ for } k \to \infty. \tag{3.38}$$

Proof. Let D be a control set with nonvoid interior and consider $x \in \mathrm{int}(D)$. By Proposition 1.3.2 (i) there are $\sigma > 0$ and $g \in \mathcal{S}_\sigma \cap \mathrm{int}(\mathcal{S})$ such that $gx = x$. Then the σ-periodic control v with $g = g(v)$ yields the σ-periodic trajectory and

$$\int_0^\sigma \Phi_v(\sigma,s)\,(Cv(s)+d)\,ds = (I - \Phi_v(\sigma,0))x.$$

Moreover, $\varphi(\cdot,x,v) \in \mathrm{int}(D)$, because $\varphi(\cdot,x,v) \in \mathcal{O}^+(x) \cap \mathcal{O}^-(x)$.

Step 1. If $1 \in \mathrm{spec}(\Phi_v(\sigma,0))$ the affine subspace $Y = x + \mathbf{E}(\Phi_v(\sigma,0);1) \subset \mathrm{int}(D)$. Indeed, the Lemma 1.4.4 shows that there is a σ-periodic solution of (1.18) starting in y if and only if $y \in Y = x + \mathbf{E}(\Phi_v(\sigma,0);1)$. Thus $g(v)y = y$, for all $y \in Y$. Proposition 1.3.2 (ii) implies that every y is in the interior of some control set, hence $Y \subset \mathrm{int}(D)$. Furthermore, Lemma 1.4.4 also yields points $x^k \in Y$ such that assertion (3.38) holds with $u := v$ and $\tau := \sigma$.

Step 2. Suppose that $1 \notin \mathrm{spec}(\Phi_v(\sigma,0))$. Lemma 1.4.2 shows that there exist $\tau_1 > 0$,

$g(u^1) \in \mathcal{S}_{\tau_1} \cap \operatorname{int}(\mathcal{S})$, and a continuous path

$$\begin{array}{rccc} p: & [0,1] & \longrightarrow & \operatorname{int}(\mathcal{S}) \\ & \alpha & \longmapsto & g(u^\alpha) \end{array}$$

where $p(0) = g(u^0) = g(v)$, with the following properties:

- u^α are τ_α-periodic controls with $\tau_0 = \sigma$;
- $g(u^\alpha) \in \mathcal{S}_{\tau_\alpha} \cap \operatorname{int}(\mathcal{S})$ for $\alpha \in [0,1]$;
- The principal fundamental solutions $\Phi_{u^\alpha}(t,s)$ of $\dot{x}(t) = A(u^\alpha(t))x(t)$ satisfy $1 \notin \operatorname{spec}(\Phi_{u^\alpha}(\tau_\alpha, 0))$ for $\alpha \in [0,1)$;
- $1 \in \operatorname{spec}(\Phi_{u^1}(\tau_1, 0))$.

The times τ_α as well as the controls $u^\alpha \in L^2([0, 2\sigma + \tau_1]\,; \mathbb{R}^m)$ depend continuously on $\alpha \in [0,1)$. Hence, $\alpha \in [0,1)$, there are unique τ_α-periodic trajectories for u^α which $x^\alpha = g(u^\alpha)x^\alpha$. By Proposition 1.3.2 (ii) there are elements in the interior of a control set. It follows that all τ_α-periodic trajectories with $\alpha \in [0,1)$ are contained in the interior of D, in particular, their initial values satisfy $x^\alpha \in \operatorname{int}(D)$.

Now consider a sequence $\alpha_k \to 1$ with $\alpha_k < 1$. If

$$\int_0^{\tau_1} \Phi_{u^1}(\tau_1, s) \left(Cu^1(s) + d \right) ds \in \operatorname{Im}(I - \Phi_{u^1}(\tau_1, 0)), \tag{3.39}$$

let with $\alpha_0 := 1$ for $k = 0, 1, 2, \ldots$

$$b_k := \int_0^{\tau_{\alpha_k}} \Phi_{u^{\alpha_k}}(\tau_{\alpha_k}, s) \left(Cu^{\alpha_k}(s) + d \right) ds, \text{ and } A_k := I - \Phi_{u^{\alpha_k}}(\tau_{\alpha_k}, 0).$$

Then $A_k x^{\alpha_k} = b_k$ and $A_k \to A_0, b_k \to b_0$ for $k \to \infty$, and $\ker A_0 = \mathbf{E}(\Phi_{u^1}(\tau_1, 0); 1)$. If x^{α_k} remains bounded, we may assume that $x^{\alpha_k} \to x^0$ for some $x^0 \in \mathbb{R}^n$ and hence $A_0 x^0 = b_0$. As in Step 1, Lemma 1.4.4 implies assertion (3.38). If x^{α_k} becomes unbounded then

$$\left\| A_0 \frac{x^{\alpha_k}}{\|x^{\alpha_k}\|} \right\| \leq \|A_0 - A_k\| + \left\| A_k \frac{x^{\alpha_k}}{\|x^{\alpha_k}\|} \right\| \leq \|A_0 - A_k\| + \frac{b_k}{\|x^{\alpha_k}\|} \to 0 \text{ for } k \to \infty,$$

and again (3.38) follows.

If (3.39) does not hold, Lemma 1.4.3 implies that, for $k = 1, 2, \ldots$, the initial values x^k of the τ_{α_k}-periodic solutions satisfy (3.38) with $u := u^1, \tau = \tau_1$. □

Next we discuss the relation of the boundary at infinity to control sets of the homogeneous part of the affine control system, motivated by the case of linear control systems exposed in the beginning of this section. If one considers affine system (3.1) far from the origin, the affine part $\sum_{i=1}^m u_i(t)c_i + d$ should become less and less relevant (note that the control range Ω is bounded). Hence the behavior might be related to the homogeneous part (3.2). We will analyze how far this is true using the compactification of the state space made possible by projecting the bilinear homogeneous system (3.24) to projective space $\mathbb{P}^n$.

First observe that every control set ${}_{\mathbb{P}}D^1 = \pi_{\mathbb{P}} D^1$ satisfies ${}_{\mathbb{P}}D^1 \cap \mathbb{P}^{n,0} = \varnothing$, since $\mathbb{P}^{n,0}$ is invariant. Recall that e defined in (3.29) denotes the diffeomorphism from $\mathbb{P}^{n-1}$ to $\mathbb{P}^{n,0}$. We note the following lemma.

Lemma 3.5.2. *Suppose that ${}_{\mathbb{P}}D^1$ is an invariant control set.*

(i) If $x \in \partial \left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0}$, then $\overline{\mathcal{O}^+(x)} \subset \partial \left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0}$.

(ii) If $e({}_{\mathbb{P}}D^{\mathrm{hom}}) \subset \mathbb{P}^{n,0}$ is a control set with $\partial \left(\pi_{\mathbb{P}} D^1\right) \cap e({}_{\mathbb{P}}D^{\mathrm{hom}}) \neq \varnothing$, then $e({}_{\mathbb{P}}D^{\mathrm{hom}}) \subset \partial \left(\pi_{\mathbb{P}} D^1\right) \cap \mathbb{P}^{n,0}$.

(iii) If $\partial \left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0} \neq \varnothing$ it contains an invariant control set $e({}_{\mathbb{P}}D^{\mathrm{hom}})$ of the system restricted to $\mathbb{P}^{n,0}$.

Proof. (i) Let $y \in \overline{\mathcal{O}^+(x)}$. Then $y \in \mathbb{P}^{n,0}$, since $\mathbb{P}^{n,0}$ is invariant and closed in $\mathbb{P}^n$. For fixed $\varepsilon > 0$ there are $T > 0$ and $u \in \mathcal{U}_{pc}$ with $d(\varphi(T, x, u), y) < \frac{\varepsilon}{2}$. Furthermore, there are $x^k \in {}_{\mathbb{P}}D^1$ with $x^k \to x$. Since ${}_{\mathbb{P}}D^1$ is an invariant control set it follows that $\varphi(T, x^k, u) \in \overline{{}_{\mathbb{P}}D^1}$, and continuous dependence on the initial values implies that $\varphi(T, x^k, u) \longrightarrow \varphi(T, x, u)$, for k large enough $d\left(\varphi(T, x^k, u), \varphi(T, x, u)\right) < \frac{\varepsilon}{2}$. Thus

$$
\begin{aligned}
d\left(\varphi\left(T, x^k, u\right), y\right) &\leq d\left(\varphi\left(T, x^k, u\right), \varphi(T, x, u)\right) + d\left(\varphi(T, x, u), y\right) \\
&\leq \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we have shown that $y \in \overline{{}_{\mathbb{P}}D^1}$ and hence $\overline{\mathcal{O}^+(x)} \subset \partial \left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0}$.

(ii) Pick $x \in \partial \left({}_{\mathbb{P}}D^1\right) \cap e({}_{\mathbb{P}}D^{\mathrm{hom}})$. Since $e({}_{\mathbb{P}}D^{\mathrm{hom}}) \subset \overline{\mathcal{O}^+(x)}$, the assertion follows from (i).

(iii) By Colonius and Kliemann [13, Theorem 3.2.8], for every point x in the compact space $\mathbb{P}^{n,0}$ there is an invariant control set $e({}_{\mathbb{P}}D^{\mathrm{hom}})$ contained in $\overline{\mathcal{O}^+(x)}$. Assertion (i) shows that every $x \in \partial \left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0}$ satisfies $\overline{\mathcal{O}^+(x)} \subset \partial \left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{n,0}$ and hence the assertion follows. □

We obtain the following consequence for invariant control sets.

Theorem 3.5.3. *Assume that the affine system (3.1) satisfies the accessibility rank condition and this system is nonhyperbolic. Suppose that D is an invariant control set.*

(i) Then the interior of D is nonvoid, the set D is unbounded in $\mathbb{R}^n$, and the boundary at infinity $\partial_\infty(D)$ contains an invariant control set $e({}_{\mathbb{P}}D^{\mathrm{hom}})$ of the system restricted to $\mathbb{P}^{n,0}$.

(ii) If the control range Ω is a compact convex neighborhood of the origin and the system on $\mathbb{P}^{n-1}$ satisfies the accessibility rank condition, then the boundary at infinity $\partial_\infty(D)$ contains the unique invariant control set $e({}_{\mathbb{P}}D^{\text{hom}})$ where ${}_{\mathbb{P}}D^{\text{hom}}$ is the unique invariant control set on $\mathbb{P}^{n-1}$.

Proof. (i) The interior of D is nonvoid, since the interior of the system semigroup is nonvoid by Theorem 1.3.1 (i). Theorem 3.5.1 shows that the set D is unbounded. It follows that $\overline{{}_{\mathbb{P}}D^1} \cap \mathbb{P}^{n,0} \neq \varnothing$, and by Proposition 3.4.2 (i) ${}_{\mathbb{P}}D^1$ is an invariant control set contained in $\mathbb{P}^{n,1}$. Hence the assertion follows from Lemma 3.5.2 (iii).

(ii) The accessibility rank condition implies that the invariant control set $e({}_{\mathbb{P}}D^{\text{hom}})$ of the system on $\mathbb{P}^{n,0}$ is unique, since this property holds by Theorem 2.2.15 (i) on $\mathbb{P}^{n-1}$. □

Our results have shown that in the nonhyperbolic case every control set with nonvoid interior is unbounded and, under the assumptions of Theorem 3.5.3 (ii), the boundary at infinity of an invariant control set of the affine control system, contains the unique invariant control set of the projectivized homogeneous part. Note, however, that there need not exist an invariant control set of the affine control system and Theorem 2.2.15 (iii) ensures that the invariant control set of the projectivized homogeneous part generates a control set in $\mathbb{R}^n \setminus \{0\}$ only if 0 is in the interior of its Floquet spectrum. In contrast, Colonius and Kliemann [13, Theorem 3.2.8] shows that there always exists an invariant control set in the compact space $\overline{\mathbb{P}^{n,1}} = \mathbb{P}^n$. Since $\mathbb{P}^{n,1}$ is invariant, there exists an invariant control set in $\mathbb{P}^{n,1}$ (it is not closed if its closure intersects $\mathbb{P}^{n,0}$).

Next we analyze the boundary at infinity of not necessarily invariant control sets. Here the chain control sets ${}_{\mathbb{P}}E_j$ of the projectivized homogeneous part will play a crucial role. Recall from Section 2.2 their relation to the control sets with nonvoid interior.

Proposition 3.5.4. *Let the assumptions of Theorem 3.5.1 be satisfied and consider a control set $D \subset \mathbb{R}^n$ with nonvoid interior of (3.1). Then there is $g(u) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S})$ such that $1 \in \text{spec}(\Phi_u(\tau,0))$ (hence 0 is a Floquet exponent) and the boundary at infinity of D satisfies*

$$e(\pi_{\mathbb{P}}(\mathbf{E}(\Phi_u(\tau,0);1)) \cap \partial_\infty(D) \neq \varnothing. \tag{3.40}$$

Proof. Theorem 3.5.1 shows that D is unbounded and that there are $g(u) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S})$ with $1 \in \text{spec}(\Phi_u(\tau,0))$ and $x^k \in \text{int}(D)$ satisfying $\|x^k\| \to \infty$ and $d(x^k, \mathbf{E}(\Phi_u(\tau,0);1)) \to 0$ for $k \to \infty$. Under the projection $\pi_{\mathbb{P}}$ to $\mathbb{P}^n$ one obtains that

$$\pi_{\mathbb{P}}\{(x,0) \mid x \in \mathbf{E}(\Phi_u(\tau,0);1)\} \subset \mathbb{P}^{n,0}.$$

Hence, in homogeneous coordinates, this leads to

$$\begin{aligned}
& d\left(\left[x_1^k:\cdots:x_n^k:1\right], \pi_{\mathbb{P}}\{(x,0)\mid x\in\mathbf{E}(\Phi_u(\tau,0);1)\}\right) \\
= \; & \inf\{d\left(\left[x_1^k:\cdots:x_n^k:1\right],[y_1:\cdots:y_n:0]\right)\mid y=(y_1,\ldots,y_n)\in\mathbf{E}(\Phi_u(\tau,0);1)\} \\
= \; & \min\left\{ d\left(\frac{x^k}{\|x^k\|},-\frac{y}{\|y\|}\right), d\left(\frac{x^k}{\|x^k\|},\frac{y}{\|y\|}\right)\middle|\; y\in\mathbf{E}(\Phi_u(\tau,0);1)\right\} \\
\leq \; & \min\left\{ \frac{2}{\|x^k\|}d\left(x^k,-y\right), \frac{2}{\|x^k\|}d\left(x^k,y\right)\middle|\; y\in\mathbf{E}(\Phi_u(\tau,0);1)\right\},
\end{aligned}$$

as $d(x^k,\mathbf{E}(\Phi_u(\tau,0);1))\to 0$ for $k\to\infty$, follow that

$$d(\left[(x_1^k:\cdots:x_n^k:1\right], \pi_{\mathbb{P}}\{(x,0)\mid x\in\mathbf{E}(\Phi_u(\tau,0);1)\}\to 0.$$

Proposition 3.4.2 (i) claim that if $x^k\in\mathrm{int}(D)$, $\|x^k\|\to\infty$ and $[y_1:\cdots:y_n:0]$ is a cluster point of $\frac{x^k}{\|x^k\|}$ so $y\in\partial_{\mathbb{P}}D\cap\mathbb{P}^{n,0}$. Furthermore,

$$e\left(\pi_{\mathbb{P}}\mathbf{E}(\Phi_u(\tau,0);1)\right)\cap\partial_\infty(D)=\pi_{\mathbb{P}}\{(x,0)\mid x\in\mathbf{E}(\Phi_u(\tau,0);1)\}\cap\partial_\infty(D)\neq\varnothing$$

□

We obtain the following result on the relation between the boundary at infinity of a control set in $\mathbb{R}^n$ and a chain control set of the homogeneous part in $\mathbb{P}^{n-1}$.

Theorem 3.5.5. *Assume that the system group of the affine control system (3.1) acts transitively on $\mathbb{R}^n$ and that the system is nonhyperbolic. Furthermore, let the control range Ω be a compact convex neighborhood of the origin. Consider a control set $D\subset\mathbb{R}^n$ with nonvoid interior of (3.1).*

(i) Then there is a chain control set ${}_{\mathbb{P}}E_j^{\mathrm{hom}}\subset\mathbb{P}^{n-1}$ such that $\partial_\infty(D)\cap e({}_{\mathbb{P}}E_j^{\mathrm{hom}})\neq\varnothing$.

If the projectivized homogeneous part on $\mathbb{P}^{n-1}$ satisfies the accessibility rank condition, the following further assertions hold.

(ii) For the control sets ${}_{\mathbb{P}}D_1^{\mathrm{hom}},\ldots,{}_{\mathbb{P}}D_{i_j}^{\mathrm{hom}}\subset{}_{\mathbb{P}}E_j^{\mathrm{hom}}$ with nonvoid interior there are points $y^i\in\pi_{\mathbb{P}}^{-1}(\overline{{}_{\mathbb{P}}D_i^{\mathrm{hom}}})$ and $\alpha_i\in\mathbb{R}$ with

$$e(\pi_{\mathbb{P}}(\alpha_1y^1+\cdots+\alpha_{i_j}y^{i_j}))\in\partial_\infty(D).$$

(iii) If the chain control set ${}_{\mathbb{P}}E_j^{\mathrm{hom}}$ in $\mathbb{P}^{n-1}$ is the closure of a control set ${}_{\mathbb{P}}D_j^{\mathrm{hom}}$ with nonvoid interior, then $0\in\overline{\Sigma_{Fl}({}_{\mathbb{P}}D_j^{\mathrm{hom}})}$ and the boundary at infinity $\partial_\infty(D)$ has a nonvoid intersection with the closure of the control set $e({}_{\mathbb{P}}D_j^{\mathrm{hom}})$.

Proof. (i) This follows from Proposition 3.5.4: Since $x=\Phi_u(\tau,0)x$ for all $x\in\mathbf{E}(\Phi_u(\tau,0);1)$ the set $\pi_{\mathbb{P}}\left(\mathbf{E}(\Phi_u(\tau,0);1)\right)$ consists of points on τ-periodic solutions for τ-periodic control u and hence is contained in a chain control set $e({}_{\mathbb{P}}E_j^{\mathrm{hom}})$ in $\mathbb{P}^{n,0}$.

(ii) By (2.18) there are control sets ${}_{\mathbb{P}}D_1^{\text{hom}}, \ldots, {}_{\mathbb{P}}D_{i_j}^{\text{hom}} \subset {}_{\mathbb{P}}E_j^{\text{hom}}$ with nonvoid interior. This implies that for $x \in {}_{\mathbb{P}}E_j^{\text{hom}}$ there are points $y^i \in \pi_{\mathbb{P}}^{-1}\left(\overline{{}_{\mathbb{P}}D_i^{\text{hom}}}\right)$, and $\alpha_i \in \mathbb{R}$ such that

$$x = \alpha_1 y^1 + \cdots + \alpha_{i_j} y^{i_j}.$$

Under the embedding e of $\mathbb{P}^{n-1}$ into $\mathbb{P}^{n,0}$ this yields a point

$$e(\pi_{\mathbb{P}}(\alpha_1 y^1 + \cdots + \alpha_{i_j} y^{i_j})) \in \partial_\infty(D).$$

This proves assertion (ii). Assertion (iii) follows from (i) and (ii). □

Under the assumptions of Theorem 3.5.5 (iii), suppose that for the control u with the property in (3.40) the Floquet exponent 0 even satisfies $0 \in \text{int}(\Sigma_{Fl}({}_{\mathbb{P}}D_j^{\text{hom}}))$. Then, by Theorem 2.2.15, the control set ${}_{\mathbb{P}}D_j^{\text{hom}}$ is the projection of a control set ${}_{\mathbb{R}}D_j^{\text{hom}}$ of the homogeneous part in $\mathbb{R}^n \setminus \{0\}$.

In order to show a partial converse of Theorem 3.5.5 we prepare the following lemma.

Lemma 3.5.6. *Suppose that $g(u) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S})$ with $1 \in \text{spec}(\Phi_u(\tau, 0))$ and*

$$\Phi_u(\tau, 0) = \exp\left(t_k A\left(u^k\right)\right) \cdots \exp\left(t_1 A\left(u^1\right)\right),$$

where $i = 1, \ldots, k$, $t_i > 0$ and $u^i \in \Omega$, hence $\tau = t_1 + \cdots + t_k$ and $u(t) = u^i$ for $t \in [t_0 + \cdots + t_{i-1}, t_0 + \cdots + t_i]$ with $t_0 = 0$.

Define for $s \in [0, t_k]$ controls u^s by $u^s(t) = u(t)$ for $t \in [0, \tau - s]$. Then $u^0 = u$ and there is $\varepsilon > 0$ such that

$$g(u^s) \in \mathcal{S}_{\tau - s} \cap \text{int}(\mathcal{S}) \text{ with } 1 \notin \text{spec}(\Phi_{u^s}(\tau - s, 0)) \quad \text{for} \quad s \in (0, \varepsilon).$$

Furthermore, $\Phi_{u^s}(\tau - s, 0) \to \Phi_u(\tau, 0)$ for $s \to 0$.

Proof. The matrices

$$\Phi_{u^s}(\tau - s, 0) = \exp((t_k - s)A(u^k)) \cdots \exp(t_1 A(u^1))$$

depend analytically on s and the same holds for $\det(I - \Phi_{u^s}(\tau - s, 0)) - 1$. Hence there are at most finitely many zeroes in $[0, t_k]$ which implies that there is $\varepsilon > 0$ with

$$1 \notin \text{spec}(\Phi_{u^s}(\tau - s, 0)) \text{ for } s \in (0, \varepsilon).$$

Taking $\varepsilon > 0$ small enough, we obtains that $g(u^s) \in \mathcal{S}_{\tau - s} \cap \text{int}(\mathcal{S})$. □

Next we formulate the announced partial converse of Theorem 3.5.5. Recall that the elements of the system semigroup ${}_{\mathbb{R}}\mathcal{S}^{\text{hom}}$ of the bilinear homogeneous system on $\mathbb{R}^n \setminus \{0\}$ can be identified with the principal fundamental solutions $\Phi_u(\tau, 0)$ for τ-periodic $u \in \mathcal{U}_{pc}$.

Theorem 3.5.7. *Assume for the homogeneous part in $\mathbb{R}^n \setminus \{0\}$ of system (3.1) that the accessibility rank condition holds. Let ${}_{\mathbb{R}}D_i^{\text{hom}}$ be a control set with nonvoid interior. Then there is a control set D with nonvoid interior of the affine system (3.1) such that its boundary at infinity satisfies, with ${}_{\mathbb{P}}D_i^{\text{hom}} \supset \pi_{\mathbb{P}}({}_{\mathbb{R}}D_i^{\text{hom}})$,*

$$\partial_\infty (D) \cap e({}_{\mathbb{P}}D_i^{\text{hom}}) \neq \varnothing. \tag{3.41}$$

Proof. Fix a point $x \in \text{int}({}_{\mathbb{R}}D_i^{\text{hom}})$. Since $\text{int}(\mathcal{S}_{\leq\tau}) \neq \varnothing$ for all $\tau > 0$ there are $\tau_0 > 0$ small enough and $u^0 \in \mathcal{U}_{pc}$ with $g(u^0) \in \mathcal{S}_{\tau_0} \cap \text{int}(\mathcal{S})$ and $x^0 := \Phi_{u^0}(\tau_0, 0)x \in \text{int}({}_{\mathbb{R}}D_i^{\text{hom}})$. Since also $\text{int}({}_{\mathbb{R}}\mathcal{S}_{\leq\tau}^{\text{hom}}) \neq \varnothing$ for all $\tau > 0$ there are $\tau_1 > 0$ small enough and $u^1 \in \mathcal{U}_{pc}$ such that the corresponding element $\Phi_{u^1}(\tau_1, 0) \in {}_{\mathbb{R}}\mathcal{S}_{\tau_1}^{\text{hom}} \cap \text{int}({}_{\mathbb{R}}\mathcal{S}^{\text{hom}})$ satisfies

$$x^1 := \Phi_{u^1}(\tau_1, 0)x^0 = \Phi_{u^1}(\tau_1, 0)\Phi_{u^0}(\tau_0, 0)x \in \text{int}({}_{\mathbb{R}}D_i^{\text{hom}}).$$

By controllability in the interior of ${}_{\mathbb{R}}D_i^{\text{hom}}$ there are $\tau_2 > 0$ and $u^2 \in \mathcal{U}_{pc}$ satisfying $\Phi_{u^2}(\tau_2, 0)x^1 = x$. Define $\tau := \tau_0 + \tau_1 + \tau_2$ and a control $u \in \mathcal{U}_{pc}$ by τ-periodic extension of

$$u(t) := \begin{cases} u^0(t) & \text{for} \quad t \in [0, \tau_0) \\ u^1(t - \tau_0) & \text{for} \quad t \in [\tau_0, \tau_0 + \tau_1) \\ u^2(t - \tau_0 - \tau_1) & \text{for} \quad t \in [\tau_0 + \tau_1, \tau_0 + \tau_1 + \tau_2) \end{cases}.$$

Then $\Phi_u(\tau, 0)x = x$, hence $1 \in \text{spec}(\Phi_u(\tau, 0))$, and

$$g(u) = g(u^2)g(u^1)g(u^0) \in \text{int}(\mathcal{S}), \quad \Phi_u(\tau, 0) = h(u) = h(u^2)h(u^1)h(u^0) \in \text{int}({}_{\mathbb{R}}\mathcal{S}^{\text{hom}}).$$

By Proposition 1.3.2 (ii) it follows that the eigenspace $\mathbf{E}(\Phi_u(\tau, 0); 1)$ of $\Phi_u(\tau, 0)$ for the eigenvalue 1 is contained in the interior of a control set in $\mathbb{R}^n \setminus \{0\}$. Since $x \in \mathbf{E}(\Phi_u(\tau, 0); 1)$ this implies that

$$\mathbf{E}(\Phi_u(\tau, 0); 1) \subset \text{int}({}_{\mathbb{R}}D_i^{\text{hom}}) \text{ and hence } \pi_{\mathbb{P}}\mathbf{E}(\Phi_u(\tau, 0); 1) \subset \text{int}({}_{\mathbb{P}}D_i^{\text{hom}}). \tag{3.42}$$

By Lemma 3.5.6 there are $g(u^s) \in \mathcal{S}_{\tau-\sigma} \cap \text{int}(\mathcal{S})$ with $1 \notin \text{spec}(\Phi_{u^s}(\tau - s, 0))$ and $\Phi_{u^s}(\tau - s, 0) \to \Phi_u(\tau, 0)$ for $s \to 0$. Proposition 1.2.1 (i) shows that there are unique $(\tau - s)$-periodic solutions of the affine equation for the $(\tau - s)$-periodic extension of u^s denoted by $\varphi(\cdot, x^s, u^s)$. By Proposition 1.3.2 (ii) they are in the interior of a control set D for the affine system (3.1).

Suppose that assumption (iii) in Lemma 1.4.3 is satisfied. Then it follows that

$$\|x^s\| \to \infty \text{ and } \frac{x^s}{\|x^s\|} \to \mathbf{E}(\Phi_u(\tau,0);1) \text{ for } s \to 0. \tag{3.43}$$

Hence, for $s \to 0$, the points $\pi_{\mathbb{P}}(x^s,1) \in \pi_{\mathbb{P}} D^1$ converge to $e\left(\pi_{\mathbb{P}}\mathbf{E}(\Phi_u(\tau,0);1)\right)$ showing that

$$\overline{\pi_{\mathbb{P}} D^1} \cap e\left(\pi_{\mathbb{P}}\mathbf{E}(\Phi_u(\tau,0);1)\right) \neq \varnothing.$$

Together with (3.42) this implies that the boundary at infinity of D satisfies (3.41). If assumption (iii) in Lemma 1.4.3 is not satisfied, Lemma 1.4.4 shows that there are $x^k \in \mathbf{E}(\Phi_u(\tau,0);1)$ with (3.43) for $k \to \infty$. This shows the assertion also in this case. □

We return to the example (3.5) (cf. Mohler [32, Example 2 on page 32]) shows that, in general, the boundary at infinity of a control set D may contain more than one control set of the projectivized homogeneous part.

Example 3.5.8. *Consider the inhomogeneous bilinear control system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2u & 1 \\ 1 & 2u \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = (A+uB)\begin{pmatrix} x \\ y \end{pmatrix} + Cu, \tag{3.44}$$

with $u \in \Omega = [-1,1]$.

We will show that there is a control set D in $\mathbb{R}^2$ such that for this extended system the projection to $\mathbb{P}^2$ yields a control set ${}_{\mathbb{P}}D^1$ which has the property that $\partial({}_{\mathbb{P}}D^1) \cap \mathbb{P}^{2,0}$ contains two control sets of the homogeneous part projectevized.

Step 1. *The eigenvalues of $A(u) = A + uB$ are $\lambda_1(u) = 2u+1 > \lambda_2(u) = 2u-1$ and $\lambda_1(-1/2) = \lambda_2(1/2) = 0$. For every $u \in \mathbb{R}$, the eigenspaces for $\lambda_1(u)$ and $\lambda_2(u)$ are $\mathbf{E}(A+uB;\lambda_1(u)) = \{(z,z)^\top \mid z \in \mathbb{R}\}$ and $\mathbf{E}(A+uB;\lambda_2(u)) = \{(z,-z)^\top \mid z \in \mathbb{R}\}$, respectively. In the northern part of the unit circle (hence in $\mathbb{P}^1$) this yields the two one-point control sets given by the equilibria for any $u \in [-1,1]$,*

$${}_{\mathbb{P}}D_1^{\text{hom}} = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\} \textit{ and } {}_{\mathbb{P}}D_2^{\text{hom}} = \left\{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}.$$

Indeed, for constant control u we have

$$exp(tA(u)) = \frac{1}{2}\begin{pmatrix} e^{t(2u+1)} + e^{t(2u-1)} & e^{t(2u+1)} - e^{t(2u-1)} \\ e^{t(2u+1)} - e^{t(2u-1)} & e^{t(2u+1)} + e^{t(2u-1)} \end{pmatrix}$$

so

$$\begin{aligned} exp(tA(u))\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^{\top} &= \frac{1}{2}\begin{pmatrix} e^{t(2u+1)}+e^{t(2u-1)} & e^{t(2u+1)}-e^{t(2u-1)} \\ e^{t(2u+1)}-e^{t(2u-1)} & e^{t(2u+1)}+e^{t(2u-1)} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{t(2u+1)}+e^{t(2u-1)}+e^{t(2u+1)}-e^{t(2u-1)} \\ e^{t(2u+1)}-e^{t(2u-1)}+e^{t(2u+1)}+e^{t(2u-1)} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{t(2u+1)} \\ e^{t(2u+1)} \end{pmatrix} \\ &= e^{t(2u+1)}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \end{aligned}$$

as $\pi_{\mathbb{P}}\left(e^{t(2u+1)}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right)=\left[\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$ *its follows that* $\left[\frac{1}{\sqrt{2}}:\frac{1}{\sqrt{2}}\right]$ *is a fixed point of the projected system of the homogeneous part of the control affine system (3.44). By the same arguments we prove that* $\left[\frac{1}{\sqrt{2}}:\frac{1}{\sqrt{2}}\right]$ *is a fixed point for this system. Hence* ${}_{\mathbb{P}}D_1^{\text{hom}}$ *and* ${}_{\mathbb{P}}D_2^{\text{hom}}$ *are two (one-point) control sets in the northern part of the unit circle.*

In $\mathbb{P}^{2,0}$ *one obtains in homogeneous coordinates*

$$e({}_{\mathbb{P}}D_1) = {}_{\mathbb{P}}D_1^0 = \left\{\left[\frac{1}{\sqrt{2}}:\frac{1}{\sqrt{2}}:0\right]\right\} \quad \textit{and} \quad e({}_{\mathbb{P}}D_2) = {}_{\mathbb{P}}D_2^0 = \left\{\left[-\frac{1}{\sqrt{2}}:\frac{1}{\sqrt{2}}:0\right]\right\}.$$

Step 2. *For* $|u|\neq\frac{1}{2}$ *the equilibria of the affine system are given by*

$$\begin{pmatrix} x_u \\ y_u \end{pmatrix} = -(A+uB)^{-1}Cu = \frac{-1}{4u^2-1}\begin{pmatrix} 2u & -1 \\ -1 & 2u \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}u = \frac{u}{4u^2-1}\begin{pmatrix} 1 \\ -2u \end{pmatrix}.$$

This leads to $y_u=-2ux_u$ *for* $|u|\neq\frac{1}{2}$. *For the asymptotics of the equilibria it follows that* $(x_u,y_u)^{\top}$ *approach the line* $\{(z,-z)^{\top}\mid z\in\mathbb{R}\}$ *for* $u\to\frac{1}{2}$ *and the line* $\{(z,z)^{\top}\mid z\in\mathbb{R}\}$ *for* $u\to-\frac{1}{2}$. *In both cases, the equilibria become unbounded. In particular, there is a connected unbounded branch of equilibria*

$$\mathcal{B}_1=\left\{\begin{pmatrix} x_u \\ y_u \end{pmatrix}\,\middle|\, u\in\left(-\frac{1}{2},\frac{1}{2}\right)\right\} \ \textit{with} \ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\in\mathcal{B}_1;$$

see Figure (3.1) (cf. also Mohler [32, Figure 2.1 on p. 33] or Rink and Mohler [35, Figure 1]). There is a single control set D_1 *containing the equilibria in* $\mathcal{B}_1$.

Step 3. *We embed the control system into a homogeneous bilinear system in* $\mathbb{R}^3$ *and project it to* $\mathbb{P}^2$. *Then the control set* $D=\{(x,y,1)\mid(x,y)\in D_1\}$ *yields a control set* ${}_{\mathbb{P}}D^1$ *in* $\mathbb{P}^{2,1}$ *given by*

$${}_{\mathbb{P}}D^1=\left\{[x:y:1]\,\middle|\,(x,y)^{\top}\in D\right\}.$$

The equilibria in $\mathcal{B}_1$ satisfy for $u \to \frac{1}{2}$ and $u \to -\frac{1}{2}$

$$\frac{(x_u, y_u)}{\|(x_u, y_u)\|} \to \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

respectively, and hence in $\mathbb{P}^2$

$$\left[\frac{x_u}{\|(x_u, y_u)\|} : \frac{y_u}{\|(x_u, y_u)\|} : \frac{1}{\|(x_u, y_u)\|}\right] \to \left[\pm\frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 0\right].$$

Consequently, both control sets $e\left({}_{\mathbb{P}}D_1^{\text{hom}}\right)$ and $e\left({}_{\mathbb{P}}D_2^{\text{hom}}\right)$ for the homogeneous part are contained in the boundary at infinity of the control set D,

$$e\left({}_{\mathbb{P}}D_1^{\text{hom}}\right) \cup e\left({}_{\mathbb{P}}D_2^{\text{hom}}\right) \subset \partial\left({}_{\mathbb{P}}D^1\right) \cap \mathbb{P}^{2,0} = \partial_\infty(D).$$

The homogeneous part of Example 3.5.8 not satisfy the accessibility rank condition in $\mathbb{P}^1$ and the control sets ${}_{\mathbb{P}}D_1$ and ${}_{\mathbb{P}}D_2$ in $\mathbb{P}^1$ have void interiors. We slightly modify this example in order to get control sets in $\mathbb{P}^1$ with nonvoid interior. Note that here an arbitrarily small perturbation suffices to change the system behavior drastically.

Example 3.5.9. *Consider for small $\varepsilon > 0$*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2u & 1 \\ 1 & (2+\varepsilon)u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = (A + uB(\varepsilon)) \begin{pmatrix} x \\ y \end{pmatrix} + Cu,$$

with $u(t) \in [-1, 1]$. We will show that there is a control set D in $\mathbb{R}^2$ such that the boundary at infinity $\partial_\infty(D) = \partial({}_{\mathbb{P}}D^1) \cap \mathbb{P}^{2,0}$ contains two control sets with nonvoid interior of the homogeneous part.

Step 1. *The eigenvalues of $A + uB(\varepsilon)$ are given by*

$$\lambda_{1,2}(u, \varepsilon) = \frac{u}{2}(4+\varepsilon) \pm \sqrt{1 + u^2\left[\frac{1}{4}(4+\varepsilon)^2 - (4+2\varepsilon)\right]}.$$

Note that $\lambda_1(u, \varepsilon) > \lambda_2(u, \varepsilon)$ for all $u \in [-1, 1]$. For $\varepsilon = 0$, it is clear that the functions $u \mapsto \lambda_{1,2}(u, 0) = 2u \pm 1$ are strictly increasing, hence this also holds for small $\varepsilon > 0$. Thus there are unique values $u^1(\varepsilon), u^2(\varepsilon) \in (-1, 1)$ with

$$\lambda_1(u^1(\varepsilon), \varepsilon) = 0 \text{ and } \lambda_2(u^2(\varepsilon), \varepsilon) = 0,$$

and $u^1(\varepsilon) \to -\frac{1}{2}$ and $u^2(\varepsilon) \to \frac{1}{2}$ for $\varepsilon \to 0$. The eigenvectors $(x, y)^\top$ satisfy $y = (\lambda_{1,2}(u, \varepsilon) - 2u)\, x$. For $\varepsilon \to 0$ and all $u \in [-1, 1]$ the eigenspace $\mathbf{E}(A + uB(\varepsilon); \lambda_i(u, \varepsilon))$ converges to the eigenspace $\mathbf{E}(A + uB(0); \lambda_i(u, 0))$. In the northern part of the unit circle (hence in $\mathbb{P}^1$) this yields the two equilibria $e_1(u, \varepsilon)$ and $e_2(u, \varepsilon)$, and the other trajectories *in* $\mathbb{P}^1$ *converge for* $t \to \infty$ *to* $e_1(u, \varepsilon)$ *and for* $t \to -\infty$ *to* $e_2(u, \varepsilon)$. *Hence there are control sets* ${}_{\mathbb{P}}D_1^{\text{hom}}$ *and* ${}_{\mathbb{P}}D_2^{\text{hom}}$ *(depending on* ε*) with nonvoid interior consisting of the equilibria* $e_1(u, \varepsilon)$ *and* $e_2(u, \varepsilon), u \in [-1, 1]$, *resp. The control set* ${}_{\mathbb{P}}D_1^{\text{hom}}$ *is invariant. One easily verifies the accessibility rank condition in* $\mathbb{P}^1$. *Since* $0 \in \text{int}(\Sigma_{Fl}({}_{\mathbb{P}}D_i^{\text{hom}}))$ *it follows that* ${}_{\mathbb{P}}D_i^{\text{hom}}$ *is the projection to* $\mathbb{P}^1$ *of a control set* ${}_{\mathbb{R}}D_i^{\text{hom}}$ *in* $\mathbb{R}^2 \setminus \{0\}$, $i = 1, 2$.

Step 2. *For* $u \neq u^1(\varepsilon), u^2(\varepsilon)$ *the equilibria are given by*

$$(A + uB(\varepsilon))\,(x_u(\varepsilon), y_u(\varepsilon))^\top = -Cu.$$

It follows that the equilibria $(x_u(\varepsilon), y_u(\varepsilon))^\top$ *approach* $\mathbf{E}(A+u^i(\varepsilon)B(\varepsilon); 0)$ *for* $u \to u^i(\varepsilon), i = 1, 2$. *In both cases, the equilibria become unbounded. In particular, there is a connected unbounded branch of equilibria*

$$\mathcal{B}_1(\varepsilon) = \left\{ (x_u(\varepsilon), y_u(\varepsilon))^\top \,\middle|\, u \in \left(u^1(\varepsilon), u^2(\varepsilon)\right) \right\}$$

and a single control set D *(again depending on* ε*) containing the equilibria in* $\mathcal{B}_1(\varepsilon)$.

Step 3. *Embedding the control system into a homogeneous bilinear system in* $\mathbb{R}^3$ *and projecting it to* $\mathbb{P}^2$ *one obtains from the control set* D *a control set in* $\mathbb{P}^{2,1}$ *given by* $\pi_{\mathbb{P}}D^1 = \left\{ [x : y : 1] \,\middle|\, (x, y)^\top \in D \right\}$. *As the equilibria* $(x_u(\varepsilon), y_u(\varepsilon))^\top \in \mathcal{B}_1$ *become unbounded for* $u \to u^i(\varepsilon)$ *they approach the eigenspace* $\mathbf{E}(A + u^i(\varepsilon)B(\varepsilon); 0)$, *respectively. It follows that*

$$e({}_{\mathbb{P}}D_1^{\text{hom}}) \cap \partial_\infty(D) \neq \varnothing \text{ and } e({}_{\mathbb{P}}D_2^{\text{hom}}) \cap \partial_\infty(D) \neq \varnothing.$$

In the following we consider chain control sets of the affine system in $\mathbb{P}^n$. The following definition is analogous to the boundary at infinity for control sets.

Definition 3.5.10. *The boundary at infinity of a chain control set* ${}_{\mathbb{P}}E$ *for the affine system (3.28) in* $\mathbb{P}^n$ *is*

$$\partial_\infty({}_{\mathbb{P}}E) := \partial({}_{\mathbb{P}}E) \cap \mathbb{P}^{n,0}.$$

This definition is similar to the boundary at infinity for control sets but it refers to chain control sets in $\mathbb{P}^n$ not requiring that they are obtained from chain control sets in $\mathbb{R}^n$.

Note that $\partial_\infty({}_{\mathbb{P}}E) \subset {}_{\mathbb{P}}E$ since chain control sets are closed.

Lemma 3.5.11. *Let* ${}_{\mathbb{P}}E$ *be a chain control set in* $\mathbb{P}^n$.

(i) If $\partial_\infty({}_{\mathbb{P}}E) \cap e({}_{\mathbb{P}}E_j^{\text{hom}}) \neq \varnothing$ *for a chain control set* ${}_{\mathbb{P}}E_j^{\text{hom}}$ *in* $\mathbb{P}^{n-1}$ *of the homogeneous part, then* $e({}_{\mathbb{P}}E_j^{\text{hom}}) \subset \partial_\infty({}_{\mathbb{P}}E)$.

(ii) If $\partial_\infty({}_{\mathbb{P}}E)$ *is nonvoid, it contains a chain control set* $e({}_{\mathbb{P}}E_j^{\text{hom}})$ *for a chain control set* ${}_{\mathbb{P}}E_j^{\text{hom}}$ *of the homogeneous part.*

Proof. (i) Recall from Proposition 3.4.2 (iii) that $e({}_{\mathbb{P}}E_j^{\text{hom}})$ is a chain control set of the system restricted to $\mathbb{P}^{n,0}$. We will show that the set

$$ {}_{\mathbb{P}}E' := {}_{\mathbb{P}}E \cup e({}_{\mathbb{P}}E_j^{\text{hom}}) $$

satisfies the properties (i) and (ii) of a chain control set in $\mathbb{P}^n$. Then the maximality property (iii) of the chain control set ${}_{\mathbb{P}}E$ implies that ${}_{\mathbb{P}}E' = {}_{\mathbb{P}}E$ showing that $e({}_{\mathbb{P}}E_j^{\text{hom}}) \subset \partial_\infty({}_{\mathbb{P}}E)$.

It is clear that ${}_{\mathbb{P}}E'$ satisfies (i), since this holds for ${}_{\mathbb{P}}E$ and $e({}_{\mathbb{P}}E_j^{\text{hom}})$. For property (ii), it suffices to consider $x \in {}_{\mathbb{P}}E$, and $y \in e({}_{\mathbb{P}}E_j^{\text{hom}})$ and $\varepsilon, T > 0$. Fix $z \in \partial_\infty({}_{\mathbb{P}}E) \cap e({}_{\mathbb{P}}E_j^{\text{hom}}) = {}_{\mathbb{P}}E \cap e({}_{\mathbb{P}}E_j^{\text{hom}})$. There are controlled (ε, T)-chains ζ_1 and ζ_2 from x to z and from z to x, respectively. For the system restricted to $\mathbb{P}^{n,0}$, there exist controlled (ε, T)-chains ζ_3 and ζ_4 from z to y and from y to z, respectively. Then the concatenations $\zeta_3 \circ \zeta_1$ and $\zeta_3 \circ \zeta_4$ are controlled (ε, T)-chains from x to y and from y to x, respectively. This concludes the proof of assertion (i).

(ii) Let $x \in \partial_\infty({}_{\mathbb{P}}E)$. Then there exists a control $u \in \mathcal{U}$ with $\varphi(t, x, u) \in \partial_\infty({}_{\mathbb{P}}E) = {}_{\mathbb{P}}E \cap \mathbb{P}^{n,0}$ for all $t \geq 0$, by property (i) of chain control sets and invariance of $\mathbb{P}^{n,0}$. Since ${}_{\mathbb{P}}E \cap \mathbb{P}^{n,0} = \partial({}_{\mathbb{P}}E) \cap \mathbb{P}^{n,0}$ is compact, it follows that the set of limit points satisfies

$$ \varnothing \neq \omega_{\mathbb{P}}(u, x) := \left\{ y = \lim_{k \to \infty} \pi_{\mathbb{P}} \varphi(t_k, x, u) \middle| \; t_k \to \infty \right\} \subset {}_{\mathbb{P}}E \cap \mathbb{P}^{n,0}. $$

Hence Colonius and Kliemann [13, Corollary 4.3.12] implies that there exists a chain control set of the system restricted to $\mathbb{P}^{n,0}$ containing $\omega_{\mathbb{P}}(u, x)$. Thus there is a chain control set ${}_{\mathbb{P}}E_j^{\text{hom}}$ in $\mathbb{P}^{n-1}$ of the homogeneous part with $\partial_\infty({}_{\mathbb{P}}E) \cap e({}_{\mathbb{P}}E_j^{\text{hom}}) \neq \varnothing$. Now the assertion follows from (i). □

The next theorem is the main result on the control sets D with nonvoid interior in $\mathbb{R}^n$ in the nonhyperbolic case. In contrast to the hyperbolic case treated in Section 3.3 D is unbounded and does not have they to unique. Using the compactification provided by $\mathbb{P}^n$, we will show that there is a single chain control set ${}_{\mathbb{P}}E$ in $\mathbb{P}^n$ containing the images of all the control sets D. The boundary at infinity $\partial_\infty({}_{\mathbb{P}}E)$ is related to the homogeneous part of the system: it contains all chain control sets $e({}_{\mathbb{P}}E_j^{\text{hom}})$ for chain control sets ${}_{\mathbb{P}}E_j^{\text{hom}}$ of the homogeneous part having nonvoid intersection with the boundary at infinity $\partial_\infty(D)$ of one of the control sets D. By Theorem 3.5.5 we know that each $\partial_\infty(D)$ has nonvoid intersection with at least one $e({}_{\mathbb{P}}E_j^{\text{hom}})$.

Theorem 3.5.12. *Assume that the affine control system (3.1) satisfies the accessibility rank condition and that it is nonhyperbolic. Furthermore, let the control range Ω be a compact convex neighborhood of the origin.*

Then there exists a single chain control set ${}_{\mathbb{P}}E$ in $\mathbb{P}^n$ containing the control sets $\pi_{\mathbb{P}} D^1$ for all control sets D with nonvoid interior in $\mathbb{R}^n$. Furthermore, the boundary at infinity

$\partial_\infty({}_{\mathbb{P}}E)$ contains all $\partial_\infty(D)$ and all chain control sets $e({}_{\mathbb{P}}E_j^{\mathrm{hom}})$, where ${}_{\mathbb{P}}E_j^{\mathrm{hom}}$ are the chain control sets in $\mathbb{P}^{n-1}$ for the homogeneous part with $\partial_\infty(D)\cap e({}_{\mathbb{P}}E_j^{\mathrm{hom}})\neq\varnothing$ for some D.

Proof. Consider control sets D_0 and D_1 in $\mathbb{R}^n$ with nonvoid interior. It suffices to show that there is a chain control set ${}_{\mathbb{P}}E$ in $\mathbb{P}^n$ containing $\pi_{\mathbb{P}}D_0^1$ and $\pi_{\mathbb{P}}D_1^1$, and that its boundary at infinity $\partial_\infty({}_{\mathbb{P}}E)$ contains all chain control sets $e({}_{\mathbb{P}}E_j^{\mathrm{hom}})$ with $\partial_\infty(D_i)\cap e({}_{\mathbb{P}}E_j^{\mathrm{hom}})\neq\varnothing$ for $i=0$ or $i=1$.

Pick $x^0\in\mathrm{int}(D_0)$ and $x^1\in\mathrm{int}(D_1)$. Then, for $i=0,1$, Proposition 1.3.2 (i) implies that there are $\tau_i>0$ and $g_i=g(u^i)\in\mathcal{S}_{\tau_i}\cap\mathrm{int}(\mathcal{S})$ with $x^i=g(u^i)x^i$. Using Lemma 1.3.4 one finds a continuous path $p:[0,1]\to\mathrm{int}(\mathcal{S})\subset\mathbb{R}^n\rtimes\mathrm{GL}(n,\mathbb{R})$ with $p(0)=g(u^0)$, $p(1)=g(u^1)$, and $p(\alpha)=g(u^\alpha)\in\mathcal{S}_{\tau_\alpha}\cap\mathrm{int}(\mathcal{S})$, where $\tau_\alpha>0$ depends continuously on α. Furthermore, $g(u^\alpha)$ depends in a piecewise analytic way on α. By Remark 1.3.5 also $u^\alpha\in L^2([0,\tau_0+\tau_1];\mathbb{R}^m)$ depends continuously on α. It follows that also $\Phi_{u^\alpha}(\tau_\alpha,0)$ depends continuously and in a piecewise analytic way on α.

If there is a point $x^\alpha\in\mathbb{R}^n$ with $x^\alpha=g(u^\alpha)x^\alpha$, Proposition 1.3.2 (ii) shows that there is a control set D^α with $x^\alpha\in\mathrm{int}(D^\alpha)$. If such a point does not exist, this means that the τ_α-periodic differential equation

$$\dot{x}(t)=A(u^\alpha(t))x(t)+Cu^\alpha(t)+d$$

has no τ_α-periodic solution. Hence, by Proposition 1.2.1 (ii), $1\in\mathrm{spec}(\Phi_{u^\alpha}(\tau_\alpha,0))$ and $\int_0^{\tau_\alpha}\Phi_{u^\alpha}(\tau_\alpha,s)\,[Cu^\alpha(s)+d]\,ds\notin\mathrm{Im}(I-\Phi_{u^\alpha}(\tau_\alpha,0))$.

Claim 1. Let $\alpha_0\in[0,1]$ be a parameter value where $1\notin\mathrm{spec}(\Phi_{u^{\alpha_0}}(\tau_{\alpha_0},0))$ and $g(u^{\alpha_0})x^{\alpha_0}=x^{\alpha_0}$ and $\alpha\mapsto g(u^\alpha)$ a continuously differentiable map in α_0. Then there is $\varepsilon>0$ such that for all $\alpha\in[0,1]$ with $|\alpha-\alpha_0|<\varepsilon$ there is x^α with $g(u^\alpha)x^\alpha=x^\alpha$ and $\alpha\mapsto x^\alpha$ is continuously differentiable.

The claim follows by the Implicit Function Theorem: The function $F(\alpha,x)=x-g(u^\alpha)x$ is continuously differentiable as a map $J\times\mathbb{R}^n\to\mathbb{R}^n$, where J is an open interval containing $[0,1]$. In fact, by (1.15)

$$\frac{\partial}{\partial x}g(u^\alpha)x=\frac{\partial}{\partial x}\varphi(\tau_\alpha,x,u^\alpha)=\Phi_{u^\alpha}(\tau_\alpha,0),$$

and $\Phi_{u^\alpha}(\tau_\alpha,0)$ depends continuously on α. One has that $F(\alpha_0,x^{\alpha_0})=0$ and

$$\frac{\partial}{\partial x}F(\alpha_0,x)=I-\Phi_{u^{\alpha_0}}(\tau_{\alpha_0},0)$$

is of full rank in $\alpha=\alpha_0$ since $1\notin\mathrm{spec}(\Phi_{u^{\alpha_0}}(\tau_{\alpha_0},0))$. Thus there is a continuously differentiable function $\alpha\mapsto x^\alpha$ defined on a neighborhood of α_0 with $0=F(\alpha,x^\alpha)=x^\alpha-g(u^\alpha)x^\alpha$ and Claim 1 follows.

Define

$$A := \{\alpha \in [0,1] \Big| \ 1 \notin \operatorname{spec}(\Phi_{u^\alpha}(\tau_\alpha, 0)) \text{ and } \alpha' \mapsto g(u^{\alpha'}) \text{ cont. differentiable in } \alpha\}.$$

There are at most only finitely many $0 \le \alpha_1 < \cdots < \alpha_r \le 1$ with $1 \in \operatorname{spec}(\Phi_{u^\alpha}(\tau_\alpha, 0))$ or $\alpha \mapsto g(u^\alpha)$ not being continuously differentiable. This holds, since $\alpha \to \det(I - \Phi_{u^\alpha}(\tau_\alpha, 0))$ and $\alpha \mapsto g(u^\alpha)$ are piecewise analytic, hence continuously differentiable for all but at most finitely many points in $[0,1]$. In particular, A consists of r' intervals, which $1 \le r' \le r+1$.

For $\alpha \in A$ there is x^α with $x^\alpha = g(u^\alpha)x^\alpha$ and $x^\alpha \in \operatorname{int}(D^\alpha)$ for some control set D^α with nonvoid interior, since $g(u^\alpha) \in \operatorname{int}(\mathcal{S})$. Claim 1 implies that for all $\alpha' \in [0,1]$ in a neighborhood of $\alpha \in A$ there is a solution of $x^{\alpha'} = g(u^{\alpha'})x^{\alpha'}$, hence $x^{\alpha'} \in \operatorname{int}(D^{\alpha'})$. For each α in an interval contained in A, x^α depends continuously on α, hence its is contained in the interior of a single control set. Denote these control sets by $D_2, \ldots, D_{r'+1}$. Since they are unbounded, $\partial_\infty(D_i)$ is nonvoid for all i. The control sets ${}_{\mathbb{P}}D_i^1$ in $\mathbb{P}^n$ are contained in chain control sets ${}_{\mathbb{P}}E_i$ and it follows that $\partial_\infty({}_{\mathbb{P}}E_i) \supset \partial_\infty(D_i)$ are nonvoid. Thus Lemma 3.5.11 (ii) implies that $\partial_\infty({}_{\mathbb{P}}E_i)$ contains a chain control set $e({}_{\mathbb{P}}E_j^{\text{hom}})$, where ${}_{\mathbb{P}}E_j^{\text{hom}}$ is a chain control set of the homogeneous part in $\mathbb{P}^{n-1}$. By Lemma 3.5.11 (i), $\partial_\infty({}_{\mathbb{P}}E_i)$ contains every chain control set of the homogeneous part that it intersects. The theorem follows from the next claim.

Claim 2. All chain control sets ${}_{\mathbb{P}}E_i, i = 2, \ldots, r'+1$, in $\mathbb{P}^n$ coincide.

We only prove this for $\alpha_i \in (0,1)$ (if $\alpha_1 = 0$ or $\alpha_{r'+1} = 1$ the proof is similar). For every point $\alpha_i \notin A$ there are control sets which we denote by D_i and D_{i+1} such that all α in a neighborhood of α_i satisfy $x^\alpha \in \operatorname{int}(D_i)$ for $\alpha < \alpha_i$ and $x^\alpha \in \operatorname{int}(D_{i+1})$ for $\alpha > \alpha_i$.

Suppose $1 \notin \operatorname{spec}(\Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0))$. Since $u^\alpha \in L^2([0, \tau_{\alpha_i} + 1], \mathbb{R}^m)$ and τ_α are continuous with respect to α, Proposition 1.2.1 (iii) implies that $x^{\alpha_0} \in \overline{D_i} \cap \overline{D_{i+1}}$. It follows that in $\mathbb{P}^n$ the intersection $\overline{\pi_{\mathbb{P}} D_i^1} \cap \overline{\pi_{\mathbb{P}} D_{i+1}^1}$ is nonvoid, showing that the chain control sets ${}_{\mathbb{P}}E_i$ and ${}_{\mathbb{P}}E_{i+1}$ containing $\pi_{\mathbb{P}} D_i^1$ and $\pi_{\mathbb{P}} D_{i+1}^1$, respectively, coincide.

It remains to consider the α_i with $1 \in \operatorname{spec}(\Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0))$. Again we have to show that the chain control sets ${}_{\mathbb{P}}E_i$ for $\alpha < \alpha_i$ and ${}_{\mathbb{P}}E_{i+1}$ for $\alpha > \alpha_i$ coincide. The projected eigenspace $\pi_{\mathbb{P}}\mathbf{E}(\Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0); 1))$ consists of points on τ_{α_i}-periodic solutions for the τ_{α_i}-periodic control u^{α_i} and hence is contained in a chain control set ${}_{\mathbb{P}}E_j^{\text{hom}}$ in $\mathbb{P}^{n-1}$. We will show that

$$e({}_{\mathbb{P}}E_j^{\text{hom}}) \subset \partial_\infty({}_{\mathbb{P}}E_i) \cap \partial_\infty({}_{\mathbb{P}}E_{i+1}),$$

which implies that ${}_{\mathbb{P}}E_i$ and ${}_{\mathbb{P}}E_{i+1}$ have nonvoid intersection and hence coincide.

First consider parameters $\beta_k \to \alpha_i, \beta_k < \alpha_i$. The points x^{β_k} with $x^{\beta_k} = g(u^{\beta_k})x^{\beta_k}$ satisfy $x^{\beta_k} \in \operatorname{int}(D_i)$.

Case 1. $\int_0^{\tau_{\alpha_i}} \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, s)\,[Cu^{\alpha_i}(s) + d]\,ds \notin \operatorname{Im}(I - \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0))$.

Lemma 1.4.3 implies that $x^{\beta_k} \in D_i$ satisfy

$$\left\|x^{\beta_k}\right\| \to \infty \text{ and } \frac{x^{\beta_k}}{\|x^{\beta_k}\|} \to \ker(I - \Phi^{\alpha_i}(\tau_{\alpha_i}, 0)) = \mathbf{E}(\Phi^{\alpha_i}(\tau_{\alpha_i}, 0); 1), \tag{3.45}$$

and it follows that

$$\varnothing \neq \partial_\infty(D_i) \cap e\left(\pi_{\mathbb{P}}\mathbf{E}(\Phi^{\alpha_i}(\tau_{\alpha_i}, 0); 1)\right) \subset \partial_\infty({}_{\mathbb{P}}E_i) \cap e({}_{\mathbb{P}}E_j^{\text{hom}}). \tag{3.46}$$

By Lemma 3.5.11 (i), $\partial_\infty({}_{\mathbb{P}}E_i)$contains $e({}_{\mathbb{P}}E_j^{\text{hom}})$.

Case 2. $\int_0^{\tau_{\alpha_i}} \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, s)\,[Cu^{\alpha_i}(s) + d]\,ds \in \operatorname{Im}(I - \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0))$.

Let, for $k = 1, 2, \ldots$

$$A_k := I - \Phi_{u^{\beta_k}}(\tau_{\beta_k}, 0), \quad b_k := \int_0^{\tau_{\beta_k}} \Phi_{u^{\beta_k}}(\tau_{\beta_k}, s)\left(Cu^{\beta_k}(s) + d\right) ds.$$

Then $x^{\beta_k} = g(u^{\beta_k})x^{\beta_k}$ implies $A_k x^{\beta_k} = b_k$ and for $k \to \infty$

$$A_k \to A_0 := I - \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0), \quad b_k \to b_0 := \int_0^{\tau_{\alpha_i}} \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, s)\,(Cu^{\alpha_i}(s) + d)\,ds.$$

If x^{β_k} remains bounded, we may assume that $x^{\beta_k} \to y^0$ for some $y^0 \in \overline{D_i} \subset \mathbb{R}^n$ and hence $A_0 y^0 = b_0$. By Lemma 1.4.4 there are $x^k \in y^0 + \mathbf{E}(\Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0))$ with $\left\|x^k\right\| \to \infty$ and $\frac{x^k}{\|x^k\|} \to \mathbf{E}(\Phi^{\alpha_i}(\tau_{\alpha_i}, 0); 1)$ and again (3.46) follows implying $e({}_{\mathbb{P}}E_j^{\text{hom}}) \subset \partial_\infty({}_{\mathbb{P}}E_i)$.

If x^{β_k} becomes unbounded, we obtain

$$\left\|A_0 \frac{x^{\beta_k}}{\|x^{\beta_k}\|}\right\| \leq \|A_0 - A_k\| + \left\|A_k \frac{x^{\beta_k}}{\|x^{\beta_k}\|}\right\| = \|A_0 - A_k\| + \frac{\|b_k\|}{\|x^{\beta_k}\|} \to 0 \text{ for } k \to \infty,$$

and (3.45) and hence (3.46) follow implying $e({}_{\mathbb{P}}E_j^{\text{hom}}) \subset \partial_\infty({}_{\mathbb{P}}E_i)$.

We have shown this inclusion using $\beta_k \to \alpha_i, \beta_k < \alpha_i$. The same arguments can be applied to parameters $\beta_k \to \alpha_i, \beta_k > \alpha_i$, showing that also for the chain control set ${}_{\mathbb{P}}E_{i+1}$ the boundary at infinity $\partial_\infty({}_{\mathbb{P}}E_{i+1})$ contains the chain control set $e({}_{\mathbb{P}}E_j^{\text{hom}})$. This proves Claim 2.

□

The following controlled linear oscillator illustrates Theorem 3.5.12.

Example 3.5.13. *Consider the affine control system (cf. [15, Example 5.17])*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1-u & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + u(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ d \end{pmatrix}, \quad u(t) \in [-\rho, \rho],$$

where $\rho \in \left(1, \frac{5}{4}\right)$ *and* $d < 1$. *The equilibria are, for* $u \in [-\rho, -1)$ *and* $u \in (-1, \rho]$,

$$\mathcal{C}_1 = \left\{ \left(\begin{array}{c} x \\ 0 \end{array} \right) \middle| \; x \in \left[\frac{d-\rho}{1-\rho}, \infty \right) \right\}, \quad \mathcal{C}_2 = \left\{ \left(\begin{array}{c} x \\ 0 \end{array} \right) \middle| \, x \in \left(-\infty, \frac{d+\rho}{1+\rho} \right] \right\},$$

respectively. The equilibria in $\mathcal{C}_1$ *are hyperbolic, since the eigenvalues of* $A(u)$ *are* $\lambda_1(u) < 0 < \lambda_2(u)$. *The equilibria in* $\mathcal{C}_2$ *are stable nodes since* $\lambda_1(u) < \lambda_2(u) < 0$. *For* $u^0 = -1$ *the matrix* $A(-1) = \left(\begin{array}{cc} 0 & 1 \\ 0 & -3 \end{array} \right)$ *has the eigenvalue* $\lambda_2(-1) = 0$ *with eigenspace* $\mathbb{R} \times \{0\}$. *There are control sets* $D_1 \neq D_2$ *containing the equilibria in* $\mathcal{C}_1$ *and* $\mathcal{C}_2$, *respectively, in the interior. For* $u^k \nearrow u^0 = -1$, *the equilibria in* D_1 *satisfy* $(x_{u^k}, 0) \to (\infty, 0)$ *and for* $u^k \searrow u^0 = -1$, *the equilibria in* D_2 *satisfy* $(x_{u^k}, 0) \to (-\infty, 0)$ *for* $k \to \infty$. *There is a single chain control set* ${}_{\mathbb{P}}E$ *in* $\mathbb{P}^2$ *containing the images of* D_1 *and* D_2, *since the eigenspace* $\mathbf{E}(e^{A(-1)\tau}; 1) = \mathbb{R} \times \{0\}$ *satisfies*

$$e(\pi_{\mathbb{P}} \mathbf{E}(e^{A(-1)\tau}; 1)) \subset \partial_\infty(D_1) \cap \partial_\infty(D_2) \quad \text{for any } \tau > 0.$$

Concerning the homogeneous part in $\mathbb{P}^1$ *the projectivized eigenspace* $\pi_{\mathbb{P}} \mathbf{E}(e^{A(-1)\tau}; 1)$ *is contained in the invariant control set* ${}_{\mathbb{P}}D_2^{\text{hom}} = \pi_{\mathbb{P}} \{ (x, \lambda_2(u)x)^\top \mid x \neq 0, u \in [-\rho, \rho] \}$ *and* ${}_{\mathbb{P}}D_2^{\text{hom}}$ *is the projection of a control set* ${}_{\mathbb{R}}D^{\text{hom}}$ *in* $\mathbb{R}^2$, *since* $0 \in \text{int}(\Sigma_{Fl}({}_{\mathbb{P}}D_2^{\text{hom}}))$.

Bibliography

[1] V. Ayala, E. Cruz, W. Kliemann, and L.R. Laura-Guarachi, *Controllability properties of bilinear systems in dimension 2*, Journal of Mathematics and Computer Science, 16 (2016), pp. 554-575.

[2] V. Ayala, and A. Da Silva, *Controllability of linear control systems on Lie groups with semisimple finite center,* SIAM J. Control Optim. 55(2) (2017) pp. 1332–1343.

[3] A. Bacciotti, and J.-C. Vivalda, *On radial and directional controllability of bilinear systems*, Systems Control Lett., 62(7) (2013), pp. 575-580.

[4] B. Bonnard, *Contrôllabilité des systèmes bilinéaires*, Math. Systems Theory, 15 (1981), pp. 79-92.

[5] B. Bonnard, V. Jurdjevic, I. Kupka, and G. Sallet, *Transitivity of families of invariant vector fields on the semi-direct product of Lie groups,* Trans. Amer. Math. Soc., 271 (1982), pp. 525-535.

[6] W. Boothby, and E.N. Wilson, *Determination of transitivity of bilinear systems,* SIAM J. Control Optim., 17 (1979), pp. 212-221.

[7] C.J. Braga Barros, and L.A.B. San Martin, *On the number of control sets on projective spaces,* Systems Control Lett., 29 (1996), pp. 21-26.

[8] C. Bruni, G. D. Pillo, and G. Koch, *Bilinear systems: an appealing class of 'nearly linear' systems in theory and applications,* IEEE Trans. Automatic Control, vol. 19, no. 4, (1974), pp. 334-348.

[9] D. Cannarsa, and M. Sigalotti, *Approximately controllable finite-dimensional bilinear systems are controllable,* Systems Control Lett., 157 (2021), Article 105028.

[10] J.W.S. Cassels, *Introduction to Diophantine Approximations*, Cambridge University Press, 1957.

[11] C. Chicone, *Ordinary Differential Equations with Applications,* Texts in Applied Mathematics, Vol. 34, Springer -Verlag 1999.

[12] F. Colonius, and W. Du, *Hyperbolic control sets and chain control sets*, J. Dynam. Control Systems 7(1) (2001), pp. 49-59.

[13] F. Colonius, and W. Kliemann, *The Dynamics of Control*, Birkhäuser 2000.

[14] F. Colonius, and W. Kliemann, *Dynamical Systems and Linear Algebra,* Graduate Studies in Mathematics, Vol. 156, Amer. Math. Soc. 2014.

[15] F. Colonius, J. Raupp, and A.J. Santana, *Control sets for bilinear and affine systems,* preprint, arXiv:2106.01204 (2021).

[16] F. Colonius, and A.J. Santana, *Topological conjugacy for affine-linear flows and control systems,* Communications on Pure and Applied Analysis, 10(3) (2011), pp. 847-857.

[17] A. Da Silva, *Controllability of linear systems on solvable Lie groups*, SIAM J. Control Optim. 54(1) (2016) 372–390.

[18] A. Da Silva, and C. Kawan, *Invariance entropy of hyperbolic control sets.* Discrete Contin. Dyn. Syst. 36(1) (2016), pp. 97-136.

[19] O. Do Rocio, L.A.B. San Martin, and A. J. Santana, *Invariant cones and convex sets for bilinear control systems and parabolic type of semigroups,* J. Dynam. Control Systems 12(3) (2006), pp. 419-432.

[20] O. Do Rocio, A.J. Santana, and M. Verdi, *Semigroups of affine groups, controllability of affine systems and affine bilinear systems in* $Sl(2,\mathbb{R}) \rtimes \mathbb{R}^2$, SIAM J. Control Optim. 48(2) (2009), pp. 1080-1088.

[21] D.L. Elliott, *Bilinear Control Systems, Matrices in Action*, Kluwer Academic Publishers, 2008.

[22] R. Engelking, *General Topology,* PWN - Polish Scientific Publishers, Warszawa, 1977.

[23] M. Firer, and O.G. Do Rocio, *Invariant control sets on flag manifolds and ideal boundaries of symmetric spaces,* J. Lie Theory, 13 (2003), pp. 463-477.

[24] J.P. Gauthier, and G. Bornard, *Controllabilité des systèmes bilinéaires*, SIAM J. Control Optim. 20(3) (1982), pp. 377-384.

[25] W. Hahn, *Stability of Motion,* Springer-Verag 1967.

[26] D. Hinrichsen, and A.J. Pritchard, *Mathematical Systems Theory, Vol. 2,* in preparation, 2022.

[27] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.

[28] V. Jurdjevic, and I. Kupka, *Control systems on semi-simple Lie groups and their homogeneous spaces,* Annales de l'Institut Fourier, tome 31, no 4 (1981), p. 151-179.

[29] V. Jurdjevic, and G. Sallet, *Controllability properties of affine systems,* SIAM J. Control Optim. 22(3) (1984), pp. 501-508.

[30] C. Kawan, *Invariance Entropy for Control Systems,* Dissertation. University of Augsburg (2010).

[31] E.B. Lee, and L. Markus, *Foundations of Optimal Control Theory,* Robert E. Krieger Publishing Company, Original Edition 1967 Reprint Edition 1986 with corrections.

[32] R.R. Mohler. *Bilinear Control Processes*, Academic Press, New York and London, 1973.

[33] J.R. Munkres. *Topology*, Prentice hall Upper Saddle River, Massachusetts, 2000.

[34] L. Perko, *Differential Equations and Dynamical Systems,* Springer, 3rd ed., 2001.

[35] R.E. Rink, and R.R. Mohler, *Completely controllable bilinear systems*, SIAM J. Control Optim. 6(3) (1968), pp. 477- 486.

[36] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos,* Taylor & Francis Inc., 2nd ed., 1998.

[37] Yu.L. Sachkov, *On invariant orthants of bilinear systems,* J. Dynam. Control Systems 4(1) (1998), pp. 137-147.

[38] L. San Martin, *Invariant control sets on flag manifolds,* Math. Control Signals Systems 6 (1993), pp. 41-61.

[39] E. Sontag, *Mathematical Control Theory*, Springer-Verlag 1998.

[40] P.L. Tchebychef, *Sur une question arithmétique, in: Oeuvres Tome I,* Imprimerie de l'Academie Impériale des Sciences, St. Petersburg, 1899, pp. 639-684.

[41] G. Teschl, *Ordinary Differential Equations and Dynamical Systems,* Graduate Studies in Math. Vol. 149, Amer. Math. Soc., 2012.

Index