STATE UNIVERSITY OF MARINGÁ

POSTGRADUATE PROGRAM IN MATHEMATICS

FELIPE GABRIEL BOGO[1]

HAMILTONIAN FORMALISM IN RIEMANNIAN GEOMETRY

MARINGÁ

2023

[1] This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001.

STATE UNIVERSITY OF MARINGÁ
POSTGRADUATE PROGRAM IN MATHEMATICS

FELIPE GABRIEL BOGO
SUPERVISOR: PROF. DR. RYUICHI FUKUOKA

HAMILTONIAN FORMALISM IN RIEMANNIAN GEOMETRY

FORMALISMO HAMILTONIANO EM GEOMETRIA RIEMANNIANA

Dissertation submitted to the Postgraduate program in Mathematics of Mathematics Departament, Exact Sciences Center of the State University of Maringá, as a requirement to obtain the title of Master in Mathematics.
Concentration area: Geometry and Topology

MARINGÁ
2023

Dados Internacionais de Catalogação na Publicação (CIP)
(Biblioteca Setorial BSE-DMA-UEM, Maringá, PR, Brasil)

Bogo, Felipe Gabriel
B675h Hamiltonian formalism in Riemannian geometry / Felipe Gabriel Bogo. -- Maringá, 2023.
98 f. : il. color.

Orientador: Prof°. Dr°. Ryuichi Fukuoka.
Dissertação (mestrado) - Universidade Estadual de Maringá, Centro de Ciências Exatas,Programa de Pós-Graduação em Matemática - Área de Concentração: Geometria e Topologia, 2023.

1. Mecânica lagrangiana. 2. Mecânica hamiltoniana. 3. Equação da geodésica. 4. Curvatura. 5. Campos de Jacobi. 6. Equações de Euler-Arnold. 7. Variedade de Finsler. I. Fukuoka, Ryuichi, orient. II. Universidade Estadual de Maringá. Centro de Ciências Exatas. Programa de Pós-Graduação em Matemática - Área de Concentração: Geometria e Topologia. III. Título.

CDD 22.ed. 516.373

Edilson Damasio CRB9-1.123

FELIPE GABRIEL BOGO

HAMILTONIAN FORMALISM IN RIEMANNIAN GEOMETRY

Dissertação apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática, Centro de Ciências Exatas da Universidade Estadual de Maringá, como parte dos requisitos necessários para a obtenção do título de Mestre em Matemática tendo a Comissão Julgadora composta pelos membros:

COMISSÃO JULGADORA:

Prof. Dr. Ryuichi Fukuoka - UEM (Presidente)
Prof. Dr. Diego Sebastian Ledesma - UNICAMP
Prof. Dr. Josiney Alves de Souza - UEM

Aprovado em: 24 de abril de 2023.
Local de defesa: Videoconferência – Google Meet (https://meet.google.com/uin-hqpq-cmt)

À minha amada esposa

Agradecimentos

Ainda que seja a tarefa mais complexa até então, me sinto honrado em dedicar meus mais profundos agradecimentos a todos e todas que fizeram parte (direta ou indiretamente) deste processo. Cada pessoa aqui mencionada teve um papel importante em minha história e, por isso, escrevo de todo meu coração.

Em primeiro lugar, gostaria de agradecer à minha esposa, Eduarda. Palavras não são suficientes para descrever meu amor e gratidão por você, entretanto, farei o possível. Te agradeço por todo apoio, dedicação, esforço e, principalmente, pelo carinho que dedicou a mim todos esses anos. Desde o primeiro dia com você aprendo a ser mais humano, observar a simplicidade da vida e a retribuir todo amor. Te agradeço por sonhar meus sonhos comigo e fazer de tudo para que eles fossem alcançados. Além disso, agradeço pela disposição de ouvir sobre matemática por incontáveis horas. Você construiu nossa família enquanto eu me dedicava a realizar meus sonhos, por isso, esta dissertação é dedicada à você.

Agradeço aos meus pais, Alessandra e Daniel, por sempre me apoiarem e fornecerem o necessário para meus estudos. Sempre me deixou feliz saber que vocês nunca questionaram minha escolha profissional e o amor de vocês foi fundamental.

Aos meus avós, Adeval (*in memoriam*), Valda, Jandira e Milton (*in memoriam*), pela educação e suporte, pelos conselhos, risadas e almoços que preenchiam meus domingos com felicidade. Também agradeço aos meus tios e tias, especialmente à Anne e ao Guilherme, que sempre me incentivaram, estiveram presentes, vibravam por mim a cada conquista e que agora trazem mais alegria para a família com o novo membro Matteo.

Aos professores presentes nesta trajetória, por todo o conhecimento compartilhado. Destaco aqui o professor Alexandre, meu ex-orientador de graduação que me deu todo apoio necessário, inclusive financeiramente, para crescer cientificamente. Aqui também aproveito para agradecer ao PET-Matemática e seus tutores pelo comprometimento com a formação dos alunos.

Ao professor Ryuichi que prontamente aceitou me orientar com muita competência e profissionalismo. As aulas de xadrez não serão esquecidas, elas me mostraram o tipo de matemático que sou e o necessário para ser mais. Agradeço imensamente pela oportunidade de trabalharmos juntos, hoje você é uma inspiração para o profissional que desejo ser no futuro.

Ao meu primeiro orientador, professor Laerte, que hoje posso chamar de amigo. Seus conselhos pessoais me tornaram uma pessoa melhor. Agradeço pelas vezes que me recebeu em sua casa para fofocar e engarrafar cerveja, foram dias extraordinários.

A todos amigos e amigas que fiz durante a graduação e mestrado, em especial, meus caros amigos Henrique, Stefane, Nathalia, Leonardo e Lucas. E, também, aos meus amigos de fora: Ana, Marcos, Allysson, Fábio, Kenia, Adriely e Wellington. Nossas confraternizações permitiram o alívio que minha mente precisava em momentos difíceis.

À CAPES pelo apoio financeiro.

À Vânia pela disposição e dedicação.

Aos meus gatos, Salem, Sofia e Mia, pelo carinho e companheirismo durante os estudos.

Este trabalho foi parcialmente realizado durante o período pandêmico mundial de Covid-19, no qual vimos tudo mudar drasticamente. Graças ao progresso técnico e científico dos últimos anos, foi possível a produção da vacina em tempo recorde, isto possibilitou a finalização deste trabalho de forma presencial. Portanto, gostaria de agradecer a todos e todas profissionais de saúde que dedicaram suas vidas à construção do conhecimento científico.

"Eu faço samba e amor até mais tarde

E tenho muito sono de manhã

Escuto a correria da cidade, que arde

E apressa o dia de amanhã"

Francisco Buarque de Hollanda

Abstract

Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold. The inner product $\langle\cdot,\cdot\rangle_p$ on the tangent space T_pM induces a isomorphism $\flat : T_pM \to T_p^*M$ between the tangent and cotangent spaces which is the Legendre transform of the Lagrangian $v \mapsto \frac{1}{2}\langle v, v\rangle_p$. This isomorphism extends to the Legendre transform between the tangent bundle TM and the cotangent bundle T^*M and we use it to transfer elements of Riemannian geometry from TM to T^*M. In this work we study the geodesic equation on T^*M through Hamiltonian formalism. We also study the Riemannian connection, curvature and Jacobi fields on T^*M.

Finally, we generalize a result contained in the paper [25] of Vladimir Kozlov. In his paper it was proved that the Lie group G with a left-invariant Riemannian metric is unimodular if and only if the Euler-Arnold flow preserve the Haar measure on the Lie algebra $\mathfrak{g}^*$. We prove that this result holds also on the Riemannian sphere of $\mathfrak{g}^*$. Afterwards we consider $\mathfrak{g}^*$ with an auxiliary Riemannian metric and we generalize the Kozlov paper for Lie groups with a left-invariant Finsler structure.

Keywords: Lagrangian mechanics, Hamiltonian mechanics, geodesic equation, curvature, Jacobi fields, Euler-Arnold equation, Finsler manifolds.

Resumo

Seja $(M, \langle \cdot, \cdot \rangle)$ uma variedade Riemanniana. O produto interno $\langle \cdot, \cdot \rangle_p$ no espaço tangente T_pM induz um isomorfismo $\flat : T_pM \to T_p^*M$ entre o fibrado tangente e cotangente o qual é a transformada de Legendre da função Lagrangiana $v \mapsto \frac{1}{2}\langle v, v \rangle_p$. Este isomorfismo se estende a transformada de Legendre entre o fibrado tangente TM e o fibrado cotangente T^*M, e é usado para transferir elementos da geometria Riemanniana de TM para T^*M. Neste trabalho estudamos as equações das geodésicas em T^*M via o formalismo Hamiltoniano. Também estudamos a conexão Riemanniana, curvatura e campos de Jacobi em T^*M.

Finalmente, nós generalizamos um resultado contido no artigo [25] do Vladimir Kozlov. Em seu artigo, foi provado que um grupo de Lie G com uma métrica Riemanniana invariante à esquerda é unimodular se, e somente se, o fluxo Euler-Arnold preserva a medida de Haar na álgebra de Lie $\mathfrak{g}^*$. Nós provamos que este resultado é válido para a esfera Riemanniana de $\mathfrak{g}^*$. Depois, consideramos $\mathfrak{g}^*$ com uma métrica Riemannian auxiliar e generalizamos o artigo do Koslov para grupos de Lie com uma estrutura de Finsler invariante à esquerda.

Palavras-chave: mecânica Lagrangiana, mecânica Hamiltoniana, equação da geodésica, curvatura, campos de Jacobi, equações de Euler-Arnold, variedade de Finsler.

Contents

Introduction

The beginning of the Riemannian geometry is dated on June 10 of 1854. Friedrich Bernhard Riemann delivered to the university of Göttingen a lecture entitled *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the Hypothesis which lie ate the Foundations of Geometry, see [35]). Riemann had been born one year after the paper *Disquisitiones generales circa superficies curves* published by Karl Friedrich Gauss in 1827, see [18]. In paper [18], Gauss starts the study of surfaces based only on the first fundamental form. In [35], Riemannian extended the notions treated by Gauss for general manifolds. Its important to note the concept of differentiable manifolds was not well established at that time.

The study of geodesics, curvatures and their relations are among the most fundamental concepts in the Riemannian geometry. Inspired by the classical mechanics, the **Lagrangian mechanics** was introduced by the mathematician and astronomer Joseph-Louis Lagrange in his 1788 work *Mécanique analytique*, see [27]. The Lagrangian formalism allow us to see geodesic as parametrized smooth curves with minimum arc length. Equivalently, geodesics is a parametrized smooth curves with null acceleration. Let $(M,\langle\cdot,\cdot\rangle)$ be a Riemannian manifold, T_xM the tangent space of M in x and T_x^*M the cotangent space of M in x. Let $TM=\{(x,y): x\in M \text{ and } y\in T_xM\}$ and $T^*M=\{(x,\xi): x\in M \text{ and } \xi\in T_xM\}$ be the tangent and cotangent bundles of M respectively. Joining the Riemannian geometry and the Lagrangian formalism, if $\gamma(t)=(\gamma^1(t),\ldots,\gamma^n(t))$ is a parametrized smooth curve on a open set $U\subset M$ that is solution of the second order differential equation on the tangent bundle TM:

$$\ddot{\gamma}^k(t)+\Gamma_{ij}^k\dot{\gamma}^i(t)\dot{\gamma}^j(t)=0 \tag{1}$$

for $k=1,\ldots,n$ then γ is a geodesic on M. (For more details see [10] and [13]).

Sir William Rowan Hamilton introduced in 1834 the **Hamiltonian mechanics** as a reformulation of Lagrangian mechanics, see [20]. The principal goal of the formalism given by Hamilton is to eventually simplify the Lagrangian formalism. As consequence, the Hamilton formalism stimulated the born of the Symplectic Geometry, that is, a smooth manifold with a non-degenerated closed 2-form. We can summarize the Hamiltonian Geometry as a geometry of symplectic manifolds with applications of momentum. The goal of Hamilton was achieved for some applications, as example via the Legendre transform $L : TM \to T^*M$ defined by $L(p,v) = (p, \langle v, \cdot \rangle_p)$ for Riemannian manifolds. The geodesic equations (1), is simplified as a first order differential equation in the cotangent bundle T^*M:

$$\begin{cases} \dot{\gamma}^j(t) = g^{ij}(\gamma(t))\xi_i(t) \\ \dot{\xi}_j(t) = \frac{1}{2}\frac{\partial g^{ik}}{\partial x^j}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^k(t) \end{cases} \tag{2}$$

where $\xi(t) = (\xi_1(t), \dots, \xi_n(t)) \in T^*_{\gamma(t)}M$ and g^{ij} is the inverse coefficients of the Riemannian metric $\langle \cdot, \cdot \rangle$. The equation (2), stimulates other questions about the simplification of invariants in Riemannian geometry, as such curvatures and Jacobi fields. In Chapter 4 we give an answer for these questions.

In 1917, Paul Finsler introduce in they doctoral thesis *Über Kurven und Flächen in allgemeinen Räumen* (On curves and surfaces in general spaces) a generalization of Riemannian metrics where the length function depends on a Minkowski norm in each tangent space, see [16]. Paul Finsler consider a version of the structure proposed by Riemann in [35] and considers a map $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) F is C^∞ on the slit tangent bundle $TM \setminus \{0\}$;

(ii) $F(x, \lambda y) = \lambda F(x,y)$ for all $\lambda > 0$;

(iii) The $n \times n$ Hessian matrix

$$(g_{ij}) = \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at every point of $TM \setminus \{0\}$.

A smooth manifold with such a structure is called **Finsler manifold**. The terminology "Finsler spaces" and "Finsler manifold" was introduced by Élie Cartan in his book *Les espaces de Finsler* (Finsler Spaces) in 1834, see [8]. In contrast to the Riemannian geometry, the g_{ij} in the Finsler structure are not parametrized by points of M, but by directions in TM. This allow us to see the Finsler manifolds as a geometry such that the inner product does not depends only of the point $p \in M$, but also depends of the direction $v \in T_pM$. The Finsler geometry has also some analogues of many natural objects in Riemannian geometry. For example, arc-length, geodesics, curvature, connections and covariant derivative, generalize for Finsler geometry. But normal coordinates do not have an analogue for Finsler manifolds, see [38]. Another difference between these two geometries is that if the Finsler structure is not Riemannian, there is no connection which is symmetrical and compatible with the metric, and there is no canonical volume element associated to the Finsler structure, see [6]. The Finsler geometry has many applications in both physics and applied sciences, see [3] [14], [23] and [26].

In 1765, Leonhard Euler showed in [15] that the motion of a rigid body in three dimensional Euclidean space is described as geodesics in the group of rotations endowed with a left-invariant metric. Vladimir Arnold rediscovered these equations in his seminal paper [5] of 1966. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The work of Arnold translated the equation given by Euler to general Lie groups, including the infinite dimensional case, endowed with a left-invariant Riemannian metric. Formally, Arnold considered the geodesic equations (2) in the Hamiltonian formalism and applied for Lie groups with a left-invariant metric, resulting in the Euler-Arnold equations

$$\dot{\xi}(t) = -\mathrm{ad}^*(\xi(t)^\sharp)\xi(t) \tag{3}$$

where $\sharp$ is the musical isomorphism induced by the Riemannian metric. In 1988, Valery Kozlov observed in [25] that the flow associated to the Euler-Arnold equations (3) preserves the volume of the Euclidean space $\mathfrak{g}^*$ if and only if G is a unimodular Lie group.

Since the Hamiltonian formalism is also equivalent to Lagrangian formalism for Finsler manifolds, this allow us to consider the Euler-Arnold equation (3) in Finsler structures and search similar properties that Kozlov found in [25] for Finsler manifolds. The flow associated to (3) remains in the spheres of $(\mathfrak{g}^*, F_*)$, where F_* is the dual norm of F, because the energy functional is preserved along this flow. In order to study Koslov's type problem in Finsler setup, we chose an auxiliary inner product $\langle \cdot, \cdot \rangle^*$ on $\mathfrak{g}^*$ in order to have a volume element on $\mathfrak{g}^*$ (Finsler structures doesn't provide natural volume elements). Our results don't depend on this choice.

In Chapter 5 we proved that

(i) Koslov's result also holds in Riemannian case when the flow is restricted to S_{F_*}.

(ii) Koslov's result can be generalized for the Finsler setting.

Now we present two recent works that are related to this work. The study of Hamiltonian formalism in Finsler manifolds can be generalized for C^0-Finsler structures, that is, a continuous function $F : TM \to \mathbb{R}$ such that $F(x,\cdot) : T_xM \to \mathbb{R}$ is a map satisfying:

(i) if $F(x,y) = 0$ then $y = 0$;

(ii) $F(x,\lambda y) = \lambda F(x,y)$, for every $\lambda \geq 0$ and $y \in T_xM$;

(iii) $F(x,y+z) \leq F(x,y) + F(x,z)$ for every $y,z \in T_xM$.

Assuming that a C^0-Finsler structure has some kind of horizontal differentiability in [37], what the authors get is that the Pontryagin's maximum principle (see [32]) of the optimal control theory can be applied and we can generalize the geodesic field of Riemannian Geometry. This study of C^0-Finsler structures looks very promising due to its differences with the Riemannian Geometry. For example, the behavior of geodesics observed in Riemann and Finsler manifolds are not satisfied in C^0-Finsler manifolds. Also, the Hamiltonian formalism its not equivalent to Lagrangian manifolds for C^0-Finsler structures and in many cases the former works better (See Subsections 9.1 and 9.2 of [37]).

In [33], Prudencio used a control theory version of the Euler-Arnold equation for C^0-Finsler manifolds using the Pontryagin's maximum principle and she studied conditions such that a solution of the Euler-Arnold equation on $\mathfrak{g}^*$ determine the extremal uniquely on $\mathfrak{g}$. In addition, she classifies the orbits of this version of Euler-Arnold equation for all three-dimensional connected Lie groups with the L^1-norm on $\mathfrak{g}$.

The text is organized as follows. In Chapter 1 we present preliminary concepts and fix some notations. In Chapter 2 we prove the Darboux theorem and study basic concepts of Symplectic Geometry. In Chapter 3 we develop the initial theory of Hamiltonian mechanics. In Section 3.4 we treat the equivalence between the Lagrangian and Hamiltonian formalisms. In Chapter 4 we study the effect of Legendre transform in the Riemannian geometrical objects such as curvatures and Jacobi fields. In Chapter 5 we generalize a Kozlov result in [25] for Riemannian spheres and for the Finsler setting.

CHAPTER 1

Preliminaries

In this chapter, we will establish some initial requisites for the dissertation. We will start with tensors in Section 1.1 and with symplectic geometry in Section 1.2. After that, the necessary objects of Riemannian geometry will be introduced in Section 1.3 and 1.4. The basics concepts of Finsler geometry will be introduced in Section 1.5.

During this work, smooth means of class C^∞ and the Einstein summation convention is in place.

1.1 Tensors

In this section, we introduce the theory of tensors on finite dimensional vector spaces and on differentiable manifolds (see [13] and [28]).

Let $\mathbb{V}$ be a n-dimensional real vector space and $\mathbb{V}^*$ its dual space.

Definition 1.1.1. *A tensor T of type (k,l) is a $(k+l)$-linear form*

$$T : \underbrace{\mathbb{V}^* \times \cdots \mathbb{V}^*}_{k \text{ times}} \times \underbrace{\mathbb{V} \times \cdots \times \mathbb{V}}_{l \text{ times}} \to \mathbb{R}$$

The addition and scalar multiplication of tensors are defined in the natural way, which gives the set of tensors of type (k,l) on $\mathbb{V}$ a vector space structure. This space will be denoted by $T^{(k,l)}(\mathbb{V})$.

Definition 1.1.2. *Let T be a tensor of type (k_1,l_1) and S a tensor of type (k_2,l_2) on $\mathbb{V}$. The* *tensor product* $T \otimes S$ *between* T *and* S *is a tensor of type* $(k_1 + k_2, l_1 + l_2)$ *defined by*

$$\begin{aligned} &T \otimes S(\alpha^1, \dots, \alpha^{k_1}, \dots, \alpha^{k_1+k_2}, v_1, \dots, v_{l_1}, \dots, v_{l_1+l_2}) \\ &\qquad := T(\alpha^1, \dots, \alpha^{k_1}, v_1, \dots, v_{l_1}) S(\alpha^{k_1+1}, \dots, \alpha^{k_1+k_2}, v_{l_1+1}, \dots, v_{l_1+l_2}). \end{aligned}$$

Definition 1.1.3. *Let* T *be a tensor of type* (k,l)*, where* $k,l > 0$*. The trace of* T *with respect to the pair* (i,j) *is the tensor of type* $(k-1, l-1)$ *defined as*

$$\begin{aligned} &(\mathrm{tr}_{(i,j)} T)(\alpha^1, \dots, \alpha^{k-1}, v_1, \dots, v_{l-1}) \\ &\qquad := T(\alpha^1, \dots, \alpha_{i-1}, e^s, \alpha^i, \dots, \alpha^k, v_1, \dots, v_{j-1}, e_s, v_j, \dots, v_l) \end{aligned}$$

where $\{e_1, \dots, e_n\}$ *is a basis of* $\mathbb{V}$ *and* $\{e^1, \dots, e^n\}$ *is its dual basis.*

Usually we denote just by tr the trace in the pair (k,l) of a (k,l)-tensor.

1.2 Smooth manifolds

The initial goal of this dissertation is to prove the Darboux theorem. This theorem states that any symplectic manifold is locally the vector space $\mathbb{R}^{2n}$ with a canonical symplectic structure. The purpose of this section is to present some notations, concepts and results required for the prove of the Darboux theorem. For more details, see [10], [28] and [39].

Let M be a n-dimensional smooth manifold. We will denote the space of smooth vector fields in M by $\mathfrak{X}(M)$. An integral curve of $X \in \mathfrak{X}(M)$ is a smooth curve $\gamma : I \to M$ such that for all $t \in I$

$$\dot{\gamma}(t) = X(\gamma(t)).$$

The vector field is complete when for any $p \in M$, exist a integral curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$.

Let $X \in \mathfrak{X}(M)$ be a complete vector field. So, for all $p \in M$, there exist an unique integral curve $\gamma_p : \mathbb{R} \to M$ such that $\gamma_p(0) = p$. Then, we can define the map

$$\begin{aligned} \phi_t : M &\to M \\ p &\mapsto \gamma_p(t) \end{aligned}$$

for all $t \in \mathbb{R}$. The application ϕ_t is not just a relation between a point $p \in M$ and its integral curve, but a diffeomorphism with inverse ϕ_{-t}. The flow generated by a vector fields allow us to define a type of derivative of Y with respect to another vector field $X \in \mathfrak{X}(M)$.

Definition 1.2.1. *Let $X \in \mathfrak{X}(M)$ and denote by ϕ_t the flow of X. For any $Y \in \mathfrak{X}(M)$ we define the **Lie derivative $\mathscr{L}_X Y$ of Y with respect to X** by*

$$(\mathscr{L}_X Y)(p) := \frac{d}{dt}\Big|_{t=0} d(\phi_{-t})_{\phi_t(p)}(Y_{\phi_t(p)}).$$

If we observe the expression of Lie bracket in local coordinates, we conclude that

$$\mathscr{L}_X Y = [X,Y]$$

for any $X,Y \in \mathfrak{X}(M)$.

Now we can extend our study to general tensor in manifolds. For that we denote by $\mathfrak{X}^*(M)$ the space of smooth 1-forms on M. A section $\alpha : M \to T^*M$ is an element of $\mathfrak{X}^*(M)$ if and only if $\alpha(X)$ is a smooth map for all $X \in \mathfrak{X}(M)$.

Definition 1.2.2. *A tensor field T of type (k,l) on a manifold M is a correspondence that for each point $p \in M$ associates a tensor $T(p) \in T^{(k,l)}(T_pM)$, i.e., given $\alpha^1,\ldots,\alpha^k \in \mathfrak{X}^*(M)$ and $Y_1,\ldots,Y_l \in \mathfrak{X}(M)$ the map*

$$T(\alpha^1,\ldots,\alpha^k,Y_1,\ldots,Y_l)$$

is a differentiable function on M and T is $\mathscr{D}(M)$-linear in each coordinate, where $\mathscr{D}(M) = \{f : M \to \mathbb{R} : f \text{ is smooth}\}$.

We denote by $T^{(k,l)}M$ the space of tensors of type (k,l) on M. Definitions 1.1.2 and 1.1.3 are preserved in the context of manifolds. We can see that any tensor T of type (k,l) on M is local and punctual, because

$$T(\alpha^1,\ldots,\alpha^n,Y_1,\ldots,Y_n)(p) = T(p)(\alpha^1(p),\ldots,\alpha^n(p),Y_1(p),\ldots,Y_n(p)).$$

.

Definition 1.2.3. *Let M and N be two smooth manifolds and $\varphi : M \to N$ be a smooth map. If T*

is a tensor of type $(0,l)$*, the* ***pullback*** $\varphi^* T$ ***of*** T ***by*** φ *is defined by*

$$(\varphi^* T)_p(v_1,\dots,v_l) = T_{\varphi(p)}(d\varphi_p(v_1),\dots,d\varphi_p(v_l)),$$

where $p \in M$ *and* $v_1,\dots,v_l \in T_pM$.

An extension of Definition 1.2.1 for tensors of type $(0,l)$ is

Definition 1.2.4. *Let* T *be a tensor of type* $(0,l)$*. We define the* ***Lie derivative of*** T ***with respect to*** X *by*

$$(\mathscr{L}_X T) := \frac{d}{dt}\Big|_{t=0} (\phi_t^* T)_p = \lim_{t\to 0} \frac{d(\phi_t)_p^*(T_{\phi_t(p)}) - T_p}{t}$$

where ϕ_t *is the flow of* X*.*

Denote by $\Omega^k(M)$ the $\mathscr{D}(M)$-module of k-forms. We will suppose knowledge of the reader about basic properties of Lie derivative of k-forms. The fundamental property of this derivative is given by the Cartan magic formula:

Theorem 1.2.5 (Cartan magic formula)**.** *For all* $\omega \in \Omega^k(M)$ *and* $X \in \mathfrak{X}(M)$*, we have that*

$$\mathscr{L}_X \omega = d i_X \omega + i_X d\omega$$

where $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ *is the interior derivative defined as* $i_X(\omega) = \omega(X,\cdot,\dots,\cdot)$*.*

Proof. See page 372 of [28]. □

This theorem is the main tool for the resolution of problems in symplectic geometry, which dates back to Élie Cartan, the mathematician who invented the modern theory of differential forms.

Let $X \in \mathfrak{X}(M)$ be a complete vector field and denote its flow by ϕ_t. Since ϕ_t is defined on $\mathbb{R}$ due to the completeness of X, the map

$$\begin{aligned} Q_t : \Omega^k(M) &\to \Omega^{k-1}(M) \\ \omega &\mapsto i_X(\phi_t^* \omega) \end{aligned}$$

is well-defined for every $t \in \mathbb{R}$. Using the Cartan magic formula

$$\frac{d}{dt}\phi_t^* \omega = Q_t(d\omega) + dQ_t(\omega)$$

where d is the exterior derivative. Define $Q(\omega) := \int_0^1 Q_t(\omega)dt$. Then

$$\phi_1^*\omega - \omega = Q(d\omega) + dQ(\omega). \tag{1.2.1}$$

This is a nice application of Cartan magic formula when we consider complete vector fields. But we can think in a family of diffeomorphism parametrized by an interval I. This give us the notion of isotopy. Namely, a smooth map $\rho : M \times I \to M$ is an isotopy if each $\rho_t := \rho(\cdot,t) : M \to M$ is a diffeomorphism with $\rho_0 = \mathrm{Id}$, where Id is the identity map. An isotopy can define one family of vector fields, which is called time dependent vector field and is given by

$$X_t(p) := \frac{d\rho_s}{ds}\Big|_{s=t}(\rho_t^{-1}(p))$$

for any $p \in M$. In other words, X_t is a family of vector fields on M such that

$$\frac{d\rho_t}{dt}(p) = X_t(\rho_t(p)).$$

The family of time dependent vector field also satisfies a similar property as (1.2.1).

Proposition 1.2.6 (Isotopy formula)**.** *Let $\rho_t : M \to M$ an isotopy. Then, exist an operator $Q : \Omega^k(M) \to \Omega^{k-1}(M)$ such that*

$$\rho_1^*\omega - \omega = dQ\omega + Qd\omega.$$

Proof. See page 40 in [10]. □

Proposition 1.2.7. *Let $\rho_t : M \to M$ be a isotopy and α_t be a family of k-forms. Then,*

$$\frac{d}{dt}\rho_t^*\alpha_t = \rho_t^*\left(\mathscr{L}_{X_t}\alpha_t + \frac{d\alpha_t}{dt}\right)$$

Proof. If $f : I \times I \to M$ is a smooth map, then

$$\frac{d}{dt}f(t,t) = \frac{d}{dx}\Big|_{x=t}f(x,t) + \frac{d}{dy}\Big|_{y=t}f(t,y).$$

Thus,

$$\begin{aligned}\frac{d}{dt}\rho_t^*\alpha_t &= \frac{d}{dx}\Big|_{x=t}\rho_x^*\alpha_t + \frac{d}{dy}\Big|_{y=t}\rho_t^*\alpha_y \\ &= \rho_t^*\mathscr{L}_{X_t}\alpha_t + \rho_t^*\left(\frac{d\alpha_t}{dt}\right) \\ &= \rho_t^*\left(\mathscr{L}_{X_t}\alpha_t + \frac{d\alpha_t}{dt}\right).\end{aligned}$$

□

1.3 Riemannian Geometry

In this section, we define the geometrical objects in Riemannian geometry that will be considered in this dissertation. More details can be found in [13].

Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold. Let ∇ be a connection on M, that is, a map $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ satisfying:

(i) $\nabla_{\varphi X+\psi Y}Z = \varphi\nabla_X Z + \phi\nabla_Y Z$,

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X(\varphi Y) = \varphi\nabla_X Y + X(\varphi)Y$,

where $X,Y,Z\in\mathfrak{X}(M)$ and $\varphi,\psi\in\mathscr{D}(M)$.

If the connection ∇ satisfies the following conditions:

(i) (Compatibility) $X\langle Y,Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X, Z\rangle$,

(ii) (Symmetry) $\nabla_X Y - \nabla_Y X = [X,Y]$,

we say that ∇ is the Levi-Civita connection or the Riemannian connection.

Definition 1.3.1. *A parametrized smooth curve $\gamma : I\to M$ is a **geodesic at** t_0 if*

$$\nabla_{\gamma'}\gamma' = 0 \tag{1.3.1}$$

*at the point t_0. If γ satisfies (1.3.1) for all $t\in I$, we say that γ is a **geodesic**.*

The **Riemannian curvature** R on M is a rule that associates for each pair of smooth vector fields $X, Y \in \mathfrak{X}(M)$ a map

$$\begin{aligned} R(X,Y) : \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ Z &\mapsto \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \end{aligned}$$

where ∇ is the Levi-Civita connection. We can observe that $R(X,Y)$ is a $\mathscr{D}(M)$-linear map and R is $\mathscr{D}(M)$-bilinear on $\mathfrak{X}(M) \times \mathfrak{X}(M)$. For $p \in M$, $R(X,Y)$ depends only on $X(p)$ and $Y(p)$ and it defines a linear map $R(X(p),Y(p)) : T_pM \to T_pM$.

Let $p \in M$, σ be a two dimensional subspace of T_pM and $\{u,v\}$ a basis of σ. We define the **sectional curvature** of σ by

$$K(\sigma) := \frac{\langle R(u,v)u,v\rangle}{|u \wedge v|^2}$$

where $|u \wedge v| = \sqrt{|u|^2|v|^2 - \langle u,v\rangle^2}$.

For any $p \in M$, let $v = z_n$ be an unit vector in T_pM. We can find an orthonormal basis $\{z_1, \ldots, z_{n-1}\}$ of T_pM that is orthogonal to v. The **Ricci and scalar curvatures** are given by:

$$\begin{aligned} Ric_p(v) &:= \frac{1}{n-1} \sum_{i=1}^{n-1} \langle R(v,z_i)v, z_i\rangle, \\ K(p) &:= \frac{1}{n} \sum_{j=1}^{n} Ric_p(z_j), \end{aligned}$$

respectively, and they doesn't depend on the choice of the orthonormal basis.

Let $\gamma : I \to M$ be a geodesic on M. We can connect the concepts of geodesics and curvature of a Riemannian manifold by the following differential equation:

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0, \tag{1.3.2}$$

where $\frac{D}{dt}$ is the covariant derivative along γ. The differential equation (1.3.2) and the solution J of (1.3.2) are called **Jacobi equation** and **Jacobi vector field**, respectively.

1.4 Riemannian Submanifolds

In this section we establish the basic theory of Riemannian submanifolds that will be used throughout the book. The main object of this section is the second fundamental form of a

Riemannian hypersurface with the induced metric. For more details, see [11] and [13].

Let M and N two differentiable manifolds with dimensions m and n respectively. A differentiable map $f : M \to N$ is called an **immersion** if the differential $df_p : T_pM \to T_{f(p)}N$ is injective for all $p \in M$. In this case, it follows that $m \leq n$. The number $k = n - m$ is called the **codimension** of f. Just for definitions, we will refer to f, or to $f(M)$, as an **immersed submanifold**

Let $\langle\cdot,\cdot\rangle_M$ and $\langle\cdot,\cdot\rangle_N$ be two Riemannian metrics on M and N respectively. An immersion $f : M \to N$ is said to be an **isometric immersion** if

$$\langle X,Y\rangle_M = f^*\langle X,Y\rangle_N \tag{1.4.1}$$

for all $X,Y \in \mathfrak{X}(M)$.

If $f : M \to N$ is an immersion and $\langle\cdot,\cdot\rangle_N$ is a Riemannian metric on N, then (1.4.1) defines a Riemannian metric on M called the **metric induced** by f with respect to which f becomes an isometric immersion.

Definition 1.4.1. *Let $(N,\langle\cdot,\cdot\rangle)$ be a Riemannian manifold, $M \subset N$ be a smooth manifold and $i : M \hookrightarrow N$ be the inclusion map. Consider M endowed with the Riemannian metric $i^*\langle\cdot,\cdot\rangle$. We say that M is a* ***Riemannian submanifold of*** N *if the topology of M is the subspace topology.*

If $f : M \to N$ is an isometric immersion, then for each $p \in M$ exist a neighborhood U of p such that $f(U)$ is a Riemannian submanifold of N. To simplify the notation, we will identify U with $f(U)$ where $U \subset M$ is an open subset and $f(U)$ is a Riemannian submanifold of N. We also identify each $X(p) \in T_pM, p \in U$ with $df_p(X(p)) \in T_{f(p)}N$. For each $p \in M$, the Riemannian metric in T_pN decomposes the tangent space in a direct sum

$$T_pN = T_pM \oplus (T_pM)^\perp$$

where $(T_pM)^\perp$ is the orthogonal complement of T_pM in T_pN.

If $X(p) \in T_pN, p \in M$ we can write

$$X(p) = X(p)^T + X(p)^\perp,\ X(p)^T \in T_pM,\ X(p)^\perp \in (T_pM)^\perp.$$

We call $X(p)^T$ by **tangential component of** $X(p)$ and $X(p)^\perp$ by **normal component of** $X(p)$.

Denote by $\overline{\nabla}$ the Riemannian connection of N. If X and Y are local vector fields on M and

$\overline{X}, \overline{Y}$ are local extensions on $\overline{N}$, then the Riemannian connection of M is given by

$$\nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T.$$

With this definitions we are ready to define the main object of this section.

Definition 1.4.2. *Let $p \in M, X(p) \in T_pM$ and $Y(p)^{\perp} \in (T_pM)^{\perp}$. The* ***second fundamental form*** *(or shape operator) of f is the map*

$$A_{Y(p)^{\perp}}(X(p)) = -(\overline{\nabla}_{X(p)} \overline{Y})^T$$

where $\overline{Y}$ is a local extension of $Y(p)^{\perp}$ normal to M.

The Definition 1.4 does not depend of the extension $\overline{Y}$. In fact, if $\overline{Y}_1$ is another extension of $Y(p)^{\perp}$, then

$$-(\overline{\nabla}_{X(p)} \overline{Y})^T - (-\overline{\nabla}_{X(p)} \overline{Y}_1)^T = (\overline{\nabla}_{X(p)} (\overline{Y}_1 - \overline{Y}))^T = 0,$$

because $\overline{Y}_1 - \overline{Y} = 0$ over a trajectory of X.

An immediate consequence of the above definition its that the second fundamental form is a symmetric operator (see [13]).

Now, considering the case of codimension 1, i.e., $f : M^n \to N^{n+1}$, $f(M) \subset N$ is called hypersurface. Let $p \in M$ and $Y(p)^{\perp} \in (T_pM)^{\perp}$ such that $||Y(p)^{\perp}|| = 1$. Since $A_{Y(p)^{\perp}} : T_pM \to T_pM$ is symmetric, exist an orthonormal basis of eigenvectors $\{e_1, \dots, e_n\}$ of T_pM with eigenvalues $k_1, \dots, k_n$. If M and N are both oriented then the vector $Y(p)^{\perp}$ is uniquely determined if we require that $\{e_1, \dots, e_n\}$ is a basis in the orientation of M and $\{e_1, \dots, e_n, Y(p)^{\perp}\}$ is a basis in the orientation of N. Therefore, we call e_i the principal directions and k_i principal curvatures of f. The symmetric functions $k_1, \dots, k_n$ are invariants of immersion. In that case, $\det(A_{Y(p)^{\perp}}) = k_1 \cdots k_n$ is called **Gauss-Kronecker curvature** of f and $\frac{1}{n}(k_1 + \cdots + k_n)$ is the **mean curvature** of f.

1.5 Finsler Manifolds

In this section we will establish some preliminary definitions about C^{∞} and C^0 Finsler geometry. Furthermore, we define left invariant Finsler structure on a Lie group.

1.5.1 Geometry on Normed Spaces

This subsection will cover the concepts of Minkowski norms and asymmetric norms. For more details about asymmetric norms, see [9], and for Minkowski norms see [6].

Throughout the text $\mathbb{V}$ denote a n-dimensional vector space over $\mathbb{R}$.

Definition 1.5.1. *A function $F = F(v) : \mathbb{V} \to \mathbb{R}$ on $\mathbb{V}$ is called a* ***Minkowski norm*** *if it has the following properties:*

(i) $F(v) \geq 0$ *for any* $v \in \mathbb{V}$*, and* $F(v) = 0$ *if and only if* $v = 0$*;*

(ii) $F(\lambda v) = \lambda F(v)$ *for any* $v \in \mathbb{V}$ *and* $\lambda > 0$*;*

(iii) F *is smooth in* $\mathbb{V} \setminus 0$*, and for any* $v \in \mathbb{V}$*, the bilinear symmetric function* $g_{(v)}$ *on* $\mathbb{V}$ *defined by*

$$g_{(v)}(v_1, v_2) := \frac{1}{2} \frac{\partial}{\partial s \partial t} [F^2(v + sv_1 + tv_2)]_{s=t=0}$$

is an inner product.

Definition 1.5.2. *An* ***asymmetric norm*** *on* $\mathbb{V}$ *is a function* $F : V \to [0, \infty)$ *satisfying the conditions:*

(i) *If* $F(v) = 0$*, then* $v = 0$*;*

(ii) $F(\lambda v) = \lambda F(v)$ *for every* $\lambda \geq 0$ *and* $v \in \mathbb{V}$*;*

(iii) $F(v + w) \leq F(v) + F(w)$*.*

In particular, a norm is clearly an asymmetric norm.

Proposition 1.5.3. *Let* F_1 *and* F_2 *be asymmetric norms on a finite dimensional vector space* $\mathbb{V}$ *over* $\mathbb{R}$*. Then there exist constants* $c, C > 0$ *such that*

$$cF_1(y) \leq F_2(y) \leq CF_1(y).$$

for every $y \in \mathbb{V}$*. Moreover, if* F *is an asymmetric norm, then* F *is continuous.*

Proof. For a proof see page 32 of [33]. □

If F is an asymmetric norm, then we can define the following subsets of $\mathbb{V}$.

$B_F(v,r) = \{w \in \mathbb{V} : F(w-v) < r\}$, open ball with center v and radius r;

$B_F[v,r] = \{w \in \mathbb{V} : F(w-v) \leq r\}$, closed ball with center v and radius r;

$S_F[v,r] = \{w \in \mathbb{V} : F(w-v) = r\}$, sphere with center v and radius r.

The next theorem ensures that the set of asymmetric norms contains the set of Minkowski norms. The notation F_{v^i} denotes the partial derivative of F with respect to v^i.

Theorem 1.5.4. *Let F be a nonnegative real-valued function on $\mathbb{V}$ with the following properties:*

1. *F is C^∞ on the punctured space $\mathbb{V} \setminus 0$;*
2. *$F(\lambda v) = \lambda F(v)$ for every $\lambda > 0$;*
3. *the $n \times n$ matrix (g_{ij}), where $g_{ij}(v) = [\frac{1}{2}F^2]_{v^i v^j}(v)$, is positive definite for every $v \neq 0$.*

Then we have the following conclusions:

(i) *(Positivity)*

$$F(v) > 0 \text{ whenever } v \neq 0;$$

(ii) *(Triangle inequality)*

$$F(v+w) \leq F(v) + F(w),$$

where the equality holds if and only if $w = \lambda v$ or $v = \lambda w$ for some $\lambda \geq 0$;

(iii) *(Fundamental inequality)*

$$w^i F_{y^i}(v) \leq F(w) \text{ for every } v \neq 0,$$

and equality holds if and only if $w = \lambda y$ for some $\lambda \geq 0$.

In particular, F is an asymmetric norm.

1.5.2 C^0-Finsler Structures

In this subsection we define Finsler and C^0-Finsler structures. By definition, we see that C^0-Finsler structures is a generalization of C^∞-Finsler structures. For more details, see [6], [17] and the Remark 2.4 in [37].

Let M be a n-dimensional manifold. Let $(x^1,\ldots,x^n): U \to \mathbb{R}^n$ be a local coordinate system on an open set $U \subset M$. The sets $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$ are the induced coordinate bases for T_pM and T_p^*M, respectively. The coordinate coordinate system $(x^1,\ldots,x^n)$ induces a local coordinates system (x^i,y^i) on TU as

$$y = y^i \frac{\partial}{\partial x^i}.$$

Similarly, $(x^1,\ldots,x^n)$ induces local coordinates (x^i,ξ_i) on T^*U as

$$\xi = \xi_i dx^i.$$

Fix a coordinate system $(x^1,x^2,\ldots,x^n)$ on an open set $U \subset M$ and the respective coordinate systems $(x^1,\ldots,x^n,y^1,\ldots,y^n)$ and $(x^1,\ldots,x^n,\xi_1,\ldots,\xi_n)$ on TU and T^*U respectively. If F is a nonnegative real-valued function on TU, the partial derivatives of F will be denoted by $F_{y^i}, F_{x^i}, F_{y^ix^j}, \ldots$, etc.

Definition 1.5.5. *A **Finsler structure** on M is a function*

$$F : TM \to [0,+\infty)$$

satisfying the following properties:

(i) *Regularity: F is C^∞ on the slit tangent bundle $TM \setminus 0$;*

(ii) *Positive homogeneity: $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$;*

(iii) *Strong convexity: the $n \times n$ Hessian matrix*

$$(g_{ij}) = \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at every point of $TM \setminus 0$.

*Given a manifold M and a Finsler structure F on M, the pair (M,F) is known as a **Finsler manifold**.*

Definition 1.5.6. *Let M be a differentiable manifold. A C^0-**Finsler structure** on M is a continuous functions $F : TM \to \mathbb{R}$ such that $F(x,\cdot) : T_xM \to \mathbb{R}$ is an asymmetric norm for every $x \in M$.*

By Theorem 1.5.4, C^0-Finsler structures generalizes Finsler structures for the setting of continuous functions. A differentiable manifold endowed with a C^0-Finsler structure is a C^0-**Finsler manifold**.

1.5.3 Left Invariant C^0-Finsler Structures in a Lie Group

In this subsection, we will restrict our manifold M to a Lie group and see what additional properties we obtain. For more details, see [34].

Definition 1.5.7. *A Lie group is a differentiable n-dimensional manifold G with a differentiable group structure, i.e., its product is differentiable.*

The map $L_g : G \to G$ defined by $L_g(h) = gh$ is called left translation on G. Denote by $\mathfrak{g}$ the space of left invariant vector fields on G, that is, if $X \in \mathfrak{X}(G)$ is such that

$$X(g) = d(L_g(e))X(e).$$

The space $\mathfrak{g}$ endowed with the usual Lie bracket on $\mathfrak{X}(G)$ is a Lie algebra, because if X and Y are left invariant vector fields, so is the bracket $[X,Y]$. The Lie algebra $\mathfrak{g}$ is called Lie algebra of G. The most important property is that $\mathfrak{g}$ is diffeomorphic to T_eG, where $e \in G$ is the identity.

If F_e is a asymmetric norm on the Lie algebra $\mathfrak{g}$ of G, we can extended it to a left invariant C^0-Finsler structure on G defining

$$F(g,v) = F_e(d(L_{g^{-1}})_g(v)), \text{ for all } (g,v) \in TG. \tag{1.5.1}$$

We can use the second condition of Definition 1.5.2 to define an asymmetric norm F_e on $\mathfrak{g}$ such that $B_{F_e}[0,1]$ is a fixed convex and compact subset of $\mathfrak{g}$ with the origin in its interior. Then we can extend that norm to a C^0-Finsler structure in G using (1.5.1).

CHAPTER 2

Symplectic Geometry

In this chapter, we prove the Darboux theorem in Section 2.1 as an application of the tools presented in Section 1.2. In Section 2.2 we study the tautological 1-form and the canonical symplectic form on the cotangent bundle T^*M of a smooth manifold M.

2.1 Darboux Theorem

This section will be used to introduce the theory of symplectic manifolds. Further ahead, we present the Moser trick, which will be used with the tools constructed in Section 1.2 to demonstrate the Darboux theorem. For more details, see [10].

Proposition 2.1.1. *Let ω be an anti-symmetric nondegenerate bilinear form on a finite real vector space $\mathbb{V}$. Then there exists a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of $\mathbb{V}$ such that*

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j)$$
$$\omega(e_i, f_j) = \delta_{ij}$$

for all $1 \le i, j \le n$. In particular, $\dim \mathbb{V}$ is even.

Proof. For a proof see [10]. □

Definition 2.1.2. *Bases like in Proposition 2.1.1 are called a* ***symplectic basis****.*

Let $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ be a symplectic basis for $\mathbb{V}$ and denote by $\{e^1, \dots, e^n, f^1, \dots, f^n\}$ its dual basis. In this context we write a symplectic form ω in $\mathbb{V}$ by

$$\omega = \sum_{i=1}^{n} e^i \wedge f^i.$$

It can be proved that

$$\begin{aligned}\omega^n &= \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}} \\ &= n!(e^{i_1} \wedge f^{i_1} \wedge \ldots \wedge e^{i_k} \wedge f^{i_k}).\end{aligned}$$

Thus, $\omega^n(e_1, f_1, \ldots, e_n, f_n) = n! \neq 0$, proving that ω is a volume form on $\mathbb{V}$.

Let M be a manifold and $\omega \in \Omega^2(M)$ be a 2-form. For any $p \in M$, the application

$$\omega_p : T_pM \times T_pM \to \mathbb{R}$$

is an anti-symmetric bilinear map. Note that ω_p varies differentiable witch respect to $p \in M$. A 2-form ω is a exact 2-form if exist a 1-form η such that

$$\omega = d\eta.$$

We say that ω is a closed 2-form if satisfies the differential equation

$$d\omega = 0.$$

Definition 2.1.3. *A 2-form ω is called symplectic form if ω is closed and the induced application*

$$\begin{aligned}\tilde{\omega}_p : T_pM &\to T_p^*M \\ w &\mapsto \omega_p(\cdot, w)\end{aligned}$$

is a linear isomorphism for each $p \in M$.

By Remark 2.1.1, the dimension of T_pM is always even. Therefore manifolds that admit a symplectic form always have even dimension.

Definition 2.1.4. *A symplectic manifold is a pair (M, ω), where M is a smooth manifold and $\omega \in \Omega^2(M)$ a symplectic form.*

As we showed, any symplectic form induces a volume form ω^n on the tangent space. Therefore, any symplectic manifold has an orientation induced by the symplectic form.

Consider the particular case when $M = \mathbb{R}^{2n}$ with linear coordinates $x^1, \ldots, x^n, y^1, \ldots, y^n$. The 2-form

$$\omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i$$

is symplectic. Indeed, taking $\alpha = -\sum_{i=1}^{n} y^i dx^i$, it follows that $d\alpha = \omega_0$, showing that ω_0 is exact. In particular, ω_0 is closed. This symplectic form is called by **canonical symplectic form of** $\mathbb{R}^{2n}$. It is easily seen

$$\left\{ \frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right\}$$

is a symplectic basis. Another example is when $M = \mathbb{S}^2(1)$ is the sphere of radius 1. Identify $T_p\mathbb{S}^2(1)$ as the space of vectors that are orthogonal to p. Consider the 2-form

$$\omega_p(u, v) = \langle p, u \times v \rangle$$

for all $u, v \in T_pM$, where $\langle \cdot, \cdot \rangle$ is the canonical inner product. Notice that the 2-form ω is non-degenerate, because for every $u \neq 0$, we can consider $v = u \times p \in T_pM$ which forms a basis of T_pM with u, where $u \times v$ is a non-zero multiple of p. In addition, since $\dim \mathbb{S}^2(1) = 2$ and $\omega \in \Omega^2(M)$ we have

$$d\omega = 0, \text{ because } d\omega \in \Omega^3(M) = \{0\}.$$

Therefore ω is a symplectic form.

The isomorphism in the category of symplectic manifolds is called symplectomorphism. A diffeomorphism $\phi : (M, \omega) \to (N, \eta)$ is a symplectomorphism if

$$\phi^* \eta = \omega.$$

The main tool for the proof of Darboux theorem is the Moser trick. Now we introduce it.

For $\omega_0, \omega_1 \in \Omega^k(M)$, we will try to build a diffeomorphism $\phi : M \to M$ such that $\phi^* \omega_1 = \omega_0$. The Moser trick is to construct ϕ as a flow of a time dependent vector field X_t on M. More precisely, for a well-behaved family of k-forms ω_t connecting ω_0 and ω_1, we try to find the time dependent vector field X_t on M such that its flow $\phi_t : M \to M$ satisfies

$$\phi_t^* \omega_t = \omega_0$$

for all $t \in I$. For this, we need to solve a differential equation. By Proposition 1.2.7:

$$\begin{aligned} 0 &= \frac{d}{dt}(\phi_t^* \omega_t - \omega_0) \\ &= \frac{d}{dt}\phi_t^* \omega_t \\ &= \phi_t^*(\mathscr{L}_{X_t}\omega_t + \dot{\omega}(t)). \end{aligned}$$

By Cartan magic formula (1.2.5),

$$0 = \phi_t^*(i_{X_t} d\omega_t + d i_{X_t}\omega_t + \dot{\omega}(t)).$$

Since ϕ_t is a diffeomorphism for all $t \in I$, ϕ_t^* is a linear isomorphism. Thus, the differential equation has a solution if and only if

$$i_{X_t} d\omega_t + d i_{X_t}\omega_t + \dot{\omega}(t) = 0$$

has a solution. So, if the equation has solution, we can find a time dependent vector field X_t such that its flow is a symplectomorphism between (M, ω_0) and (M, ω_1) for all $t \in I$.

Definition 2.1.5. *A manifold M is said* ***closed manifold*** *if M is a compact topological space without boundary.*

The first application of the Moser trick is a classification of maps that preserve volume forms in a closed and orientable manifold.

Theorem 2.1.6. *Let M be a orientable, closed manifold and ω_0, ω_1 be two volume forms on M. Then there exist a diffeomorphism $\phi : M \to M$ such that $\phi^*\omega_1 = \omega_0$ if and only if*

$$\int_M \omega_0 = \int_M \omega_1.$$

Proof. If there exist a diffeomorphism $\phi : M \to M$ such that $\phi^*\omega_1 = \omega_0$, then,

$$\int_M \omega_0 = \int_M \phi^*\omega_1 = \int_{\phi(M)} \omega_1 = \int_M \omega_1.$$

Reciprocally, since M is closed and orientable, it follows that

$$H^n_{dR}(M) = \mathbb{R}$$

where H^n_{dR} is the de Rham cohomology. So, if

$$\int_M \omega_0 = \int_M \omega_1,$$

it follows that

$$\int_M [\omega_0 - \omega_1] = 0.$$

Therefore, $\omega_0 - \omega_1 = 0 \in H^n_{dR}(M)$. Thus, exist $\beta \in \Omega^{n-1}(M)$ such that

$$\omega_1 - \omega_0 = d\beta.$$

Now, consider the following family of k-forms

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

We are now going to show that ω_t is a family of volume forms. Indeed, as ω_0 and ω_1 are volume forms on the compact smooth manifold M, there exist $f \in \mathscr{D}(M)$ such that $f(p) \neq 0$ for all $p \in M$ and $\omega_1 = f\omega_0$. Since

$$\int_M \omega_0 = \int_M \omega_1 = \int_M f\omega_0,$$

it follows that $f > 0$. Then,

$$\begin{aligned}
\omega_t &= (1-t)\omega_0 + t\omega_1 \\
&= \omega_0 - t\omega_0 + tf\omega_0 \\
&= (1-t+tf)\omega_0,
\end{aligned}$$

what proves that ω_t is a family of volume forms because $\omega_0 \neq 0$ for all $p \in M$. Applying the Moser trick in this family, we have

$$\begin{aligned}
0 &= i_{X_t} d\omega_t + di_{X_t}\omega_t + \dot{\omega}(t) \\
&= di_{X_t}\omega_t + (\omega_1 - \omega_0) \\
&= di_{X_t}\omega_t + d\beta.
\end{aligned}$$

As ω_t is non-null for all $t \in I$, the differential equation $-d\beta = di_{X_t}\omega_t$ has solution X_t for all

$t \in I$, ensuring the existence of a time dependent vector field X_t with flow $\phi_t : M \to M$ such that

$$\phi_t^* \omega_t = \omega_0.$$

Taking $t = 1$, we have the desired. □

As a corollary, we can classify the closed symplectic surfaces of dimension 2. We call by symplectic area of a symplectic 2-manifold (M, ω) the integral

$$\text{Area}(\omega) := \int_M \omega.$$

Theorem 2.1.7 (Classification of closed Symplectic surfaces)**.** *Let* $(M_1, \omega_1), (M_2, \omega_2)$ *be two closed symplectic surfaces. These surfaces are symplectomorphic if and only if they have the same genus and symplectic area.*

Proof. We know that two closed and orientable surfaces are diffeomorphic if and only if M_1 and M_2 has the same genus and Theorem 2.1.6 guarantees that this diffeomorphism is a symplectomorphism if and only if they has the same symplectic area. □

Let M be a n-dimensional manifold and N be a k-dimensional submanifold of M by the inclusion

$$i : N \hookrightarrow M.$$

We can see the tangent space T_pN as a subspace of the tangent space T_pM for all $p \in N$ by the linear inclusion $di_p : T_pN \hookrightarrow T_pM$. The quotient space $\mathscr{N}_pN := T_pM/T_pN$ is a $(n-k)$-dimensional vector space called **normal space to** N **in** p. The **normal bundle of** N is the bundle

$$\mathscr{N}N := \{(p, v) : p \in N \text{ and } v \in \mathscr{N}_pN\}.$$

The set $\mathscr{N}N$ has structure of vector bundle over N of rank $n-k$ by the natural projection. So, $\mathscr{N}N$ is an n-dimensional smooth manifold. The null section of $\mathscr{N}N$ is given by

$$\begin{aligned} i_0 : N &\hookrightarrow \mathscr{N}N \\ p &\mapsto (p, 0) \end{aligned}$$

and it is an embedding of N as a closed submanifold of $\mathscr{N}N$. A neighborhood U_0 of the null section N in $\mathcal{N}N$ is called **convex** if the intersection $U_0 \cap \mathcal{N}_p N$ for all $p \in M$ are convex. With this notations, we can enunciate the following result.

Theorem 2.1.8 (Theorem of tubular neighborhood)**.** *Let M be a n-dimensional smooth manifold and N a k-dimensional submanifold of M. Exist a convex neighborhood U_0 of N on $\mathcal{N}N$, a neighborhood U of N on M and a diffeomorphism $\varphi : U_0 \to U$ such that the diagram*

$$\begin{tikzcd} U_0 \subset \mathcal{N}N \arrow[rr, "\varphi"] & & U \subset M \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes.

Proof. For proof, see page 37 in [10]. □

The following proposition is an application of Theorem 2.1.8

Proposition 2.1.9 (Poincaré Lemma)**.** *Let U be a tubular neighborhood of N in M. If $\omega \in \Omega^k(U)$ is closed in U and $i^*\omega = 0$, then ω is exact. Moreover, we can take $\mu \in \Omega^{k-1}(U)$ such that $\omega = d\mu$ and $\mu_p = 0$ for all $p \in N$.*

Proof. Since U is a tubular neighborhood of N, exist a convex neighborhood U_0 of N in $\mathcal{N}N$ and a diffeomorphism

$$\varphi : U_0 \to U.$$

Define the isotopy

$$\begin{aligned} \rho_t : U_0 &\to U \\ (p,v) &\mapsto \varphi(p,tv). \end{aligned}$$

Since U_0 is a convex subset, this isotopy is well-defined. We know that the operator

$$Q(\omega) = \int_0^1 \rho_t^*(i_{X_t}\omega)dt$$

satisfies

$$\rho_1^*(\omega) - \rho_0^*(\omega) = dQ(\omega) + Q(d\omega),$$

see page 40 of [10]. As ω is closed, it follows that

$$\omega = dQ(\omega),$$

because $i^*\omega = 0$. Define $\mu = Q(\omega)$. Then $d\mu = dQ(\omega) = \omega$. Now, we need to show that $\mu_{(p,0)} = 0$ for all $p \in N$. In fact, $\rho_t(p,0) = \varphi(p,0)$ for all $p \in N$ and $t \in I$. Thus,

$$\frac{d}{dt}\rho_t(p,0) = 0,$$

implying that

$$(d\rho_t)_{(p,0)}\left(\frac{d}{dt}\right) = 0.$$

Thus, $\mu_{(p,0)} = Q(\omega)(p,0) = 0$. □

The next Lemma will be useful for the proof of Weinstein's theorem.

Lemma 2.1.10. *Let X_t be a time dependent vector field on M. Suppose that $X_t|_{p_0} = 0$ for all $t \in I$ where I is a closed interval. Then there exists a neighborhood U of p_0 on M such that ρ_t is defined for all $x \in U$ and $t \in I$.*

Idea of proof. In local coordinates with $p_0 = 0$,

$$X_t = f^i(t,x^1,\dots,x^n)\frac{\partial}{\partial x^i}$$

and there exist $\tilde{C} > 0$ and a neighborhood U of p_0 such that

$$|f^i(t,x^1,\dots,x^n)| \leq \tilde{C}||(x^1,\dots,x^n)||$$

for every $(t,x^1,\dots,x^n) \in I \times U$. Here $||\cdot||$ stands for the Euclidean metric. Therefore there exist $C > 0$ such that

$$||X_t(x)|| \leq C||(x^1,\dots,x^n)||$$

for every $(t,x) \in I \times U$. This means that if $\rho(t)$ is a trajectory of X_t, then

$$||\rho(t)|| \leq ||\rho(0)||e^{Ct}.$$

Therefore if we choose a sufficiently small neighborhood of p_0, then ρ_t is well defined. □

Theorem 2.1.11 (Weinstein's Theorem)**.** *Let M be a smooth manifold and N be a submanifold of M. If ω_0, ω_1 are two symplectic forms of M such that*

$$\omega_0|_N = \omega_1|_N,$$

then there exist neighborhoods U_0, U_1 of N on M and a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(p) = p$ for all $p \in N$, the diagram

$$\begin{tikzcd} U_0 \arrow[rr, "\varphi"] & & U_1 \\ & N \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes and $\varphi^* \omega_1 = \omega_0$.

Proof. By hypothesis, $\omega_0 - \omega_1$ is closed and $i^*(\omega_0 - \omega_1) = 0$. By Poincaré Lemma 2.1.9, there exists a neighborhood U of N and $\mu \in \omega^1(U)$ such that $\omega_1 - \omega_0 = d\mu$ and $\mu|_N = 0$ for all $p \in N$. Let

$$\begin{aligned} \omega_t &= t\omega_1 + (1-t)\omega_0 \\ &= \omega_0 + t(\omega_1 - \omega_0) \\ &= \omega_0 + td\mu \end{aligned} \tag{2.1.1}$$

a family of 2-forms on U. We will show that the family of 2-forms ω_t is non-degenerated on N. Indeed, consider the map

$$\begin{aligned} \psi : N \times [0,1] &\to \mathbb{R} \\ (p,t) &\mapsto \det \omega_t(p) \end{aligned}$$

This function is non-null for all $p \in N$, because

$$\begin{aligned} \psi(p,t) &= \det \omega_t(p) \\ &= \det(\omega_0 + td\mu)(p) \\ &= \det \omega_0(p) \end{aligned}$$

is non-degenerated for all $t \in I$ by (2.1.1), what implies that ψ is non-null in a neighborhood of $\{p\} \times I$, which can be chosen as $U \times I$ because I is compact. Therefore, we have a neighborhood U for N such that ω_t is symplectic for all $t \in I$.

By the Moser trick, we can define X_t as solution of $i_{X_t}\omega_t = -\mu$ because ω_t is a family of symplectic forms. Since $\mu|_N = 0$, it follows that

$$i_{X_t}\omega_t = \omega(X_t, \cdot) = 0, \tag{2.1.2}$$

so $X_t|_N = 0$ and $\rho_t|_N = \mathrm{Id}_N$, where ρ_t its the time dependent map induced by X_t. By the previous Lemma, there exists a neighborhood U_0 of N such that $U_0 \subset U$ and ρ_t is defined for all $t \in I$. Therefore, putting $\rho_1 = \varphi$ and $U_1 = \rho_1(U_0)$, we have the desired. □

As a particular case of Weinstein theorem, we have the principal theorem of the section

Theorem 2.1.12 (Darboux Theorem)**.** *Let (M,ω) be a symplectic manifold of dimension $2n$. For all $p \in M$ exists a neighborhood U of p in M and an open subset U_0 of $\mathbb{R}^{2n}$ such that (U,ω) is symplectomorphic to (U_0,ω), where ω_0 is the canonical symplectic form of $\mathbb{R}^{2n}$.*

Proof. Let $\{x^{1'},\ldots,x^{n'},y^{1'},\ldots,y^{n'}\}$ be a symplectic basis for (T_pM,ω_p). Extended this in a neighborhood U' of p in M. On U' has two simplectic forms, namely $\omega_0 = \omega$ and

$$\omega_1 = \sum_{i'=1}^{n} dx^{i'} \wedge dy^{i'}.$$

Now, apply the Weinstein's theorem on $N = \{p\}$ and $M = U'$. Therefore, exist a neighborhoods U_0 and U_1 of p in U', a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(p) = p$ and

$$\varphi^*(\sum_{i'=1}^{n} dx^{i'} \wedge dy^{i'}) = \omega.$$

Putting $x^i = x^{i'} \circ \varphi$ and $y^i = y^{i'} \circ \varphi$, we finish the proof. □

The proof of Darboux Theorem relies on the fact that a symplectic form ω is closed. If ω is not closed we cannot guarantee this theorem. Indeed, if ω is not closed then the Moser trick in the Weinstein's theorem is given by

$$i_{X_t} d\omega_t + d i_{X_t}\omega_t = -\mu \tag{2.1.3}$$

If X_t is a solution of (2.1.3), then we cannot guarantee that $X_t|_N = 0$ as in (2.1.2) and apply the Lemma 2.1.10.

While Riemannian geometry is about non-degenerate symmetric bilinear forms on tangent spaces, symplectic geometry is about non-degenerate anti-symmetric bilinear forms on tangent spaces with an additional condition described by the differential equation $d\omega = 0$. We can impose a differential equation for the metric on a Riemannian manifold. In fact, not all Riemannian manifolds are isometric to the Euclidean space with the canonical metric but Riemannian manifolds with curvature tensor identically null are locally isometric to the Euclidean space. Therefore, this condition in the Riemannian case can be compared to the closedness of a symplectic structure on a manifold.

2.2 Symplectic Form on the Cotangent Bundle

Now we will restrict our study to the cotangent bundle. In this section, we will put a specific symplectic form in the cotangent bundle and study their properties. In short, we will induce a symplectomorphism between cotangent bundles whenever exists a diffeomorphism between manifolds. For more details, see [10].

Let M be an n-dimensional smooth manifold. Let $\pi : T^*M \to M$ be the natural projection on M. The **tautological** 1**-form on** T^*M can be defined at $p = (x, \xi)$ by

$$\alpha_p = (d\pi_p)^*\xi \in T_p^*(T_x^*M) \tag{2.2.1}$$

with $(d\pi_p)^*$ being the transpose of $d\pi_p$, i.e., $(d\pi_p^*)\xi = \xi \circ d\pi_p$. Equivalently,

$$\alpha_p(v) = (d\pi_p)^*(\xi)(v) = \xi(d\pi_p(v)), \text{ for all } v \in T_p(T^*M).$$

Clearly, the definition of equation (2.2.1) does not depend of the choice of coordinates. Now, let $(x^1, \dots, x^n) : U \to \mathbb{R}^n$ be a local coordinate system centered in $x \in M$ with the corresponding natural coordinates $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$ in cotangent space given by $\xi_i dx^i \mapsto (x^1(x), \dots, x^n(x), \xi_1,$ $\dots, \xi_n)$ centered in $p = (x, \xi)$. Take $v = v^i \frac{\partial}{\partial x^i} + \tilde{v}_j \frac{\partial}{\partial \xi_j} \in T_p(T^*M)$. Notice that

$$d\pi_p\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}$$

and

$$d\pi_p\left(\frac{\partial}{\partial \xi_j}\right) = 0.$$

Thus,

$$\begin{aligned} \alpha_p(v) &= \xi(d\pi_p)(v) \\ &= \xi_i (dx^i)_x \left(v^i \frac{\partial}{\partial x^i}\right). \end{aligned}$$

Therefore, locally $\alpha = \xi_i dx^i$.

The **symplectic canonical form ω on** T^*M is defined by

$$\omega = d\alpha.$$

And locally it is given by $\omega = d\xi_i \wedge dx^i$.

Observe that the tautological 1-form is uniquely determined by the following property: For all 1-form $\mu : M \to T^*M$ we have that

$$\mu^*\alpha = \mu.$$

Indeed, take $p = (x, \mu_x) \in T^*M$. Then,

$$\begin{aligned}(\mu^*\alpha)_p &= (d\mu)_x^*\alpha_p \\ &= (d\mu)_x^*(d\pi_p)^*\mu_x \\ &= (d\underbrace{(\pi \circ \mu_x)}_{\mathrm{Id}_M})_x^*\mu_x \\ &= \mu_x.\end{aligned}$$

Now, suppose the existence of another 1-form β on T^*M satisfying the same property. Thus,

$$\mu^*\beta = \mu = \mu^*\alpha.$$

Therefore, $\mu^*(\beta - \alpha) = 0$ for all $\mu \in \mathfrak{X}^*(M)$. So for any $v \in T_xM$,

$$0 = (\beta - \alpha)(d\mu)_x(v).$$

For each $p = (x, \xi) \in T^*M$, the set

$$\{(d\mu)_x v : \mu \in \mathfrak{X}^*(M), \mu_x = \xi \text{ and } v \in T_xM\}$$

span $T_p(T^*M)$, so we conclude that $\beta = \alpha$.

Let M and N be two manifolds of dimension n with tautological 1-forms $\alpha_1 : T^*M \to T(T^*M)$ and $\alpha_2 : T^*N \to T(T^*N)$, respectively and $f : M \to N$ be a diffeomorphism. We are now going to show that exist a natural diffeomorphism

$$f_\sharp : T^*M \to T^*N$$

that **lifts** f. In fact, take $p_1 = (x_1, \xi_1) \in T^*M$. Define,

$$f_\sharp(p_1) = p_2 = (x_2, \xi_2)$$

where

$$\begin{cases} x_2 = f(x_1) \in N \\ \xi_1 = (df_{x_1})^* \xi_2 \iff \xi_2 = ((df_{x_1})^*)^{-1} \xi_1 \end{cases}$$

and $(df_{x_1})^* : T^*_{x_2}N \to T^*_{x_1}M$ is a linear isomorphism. Notice that $f_\sharp$ is a diffeomorphism between the vector bundles T^*M and T^*N. Indeed, $(x_1, \xi) \mapsto ((df_{x_1})^*)^{-1}\xi_1$ is smooth because of the smoothness of f. Therefore $f_\sharp$ is smooth. The smoothness of $(f_\sharp)^{-1} = (f^{-1})_\sharp$ follows analogously.

Proposition 2.2.1. *The lift $f_\sharp$ of a diffeomorphism $f : M \to N$ pull back the tautological 1-form α_2 of T^*N to the tautological 1-form α_1 on T^*M, i.e.,*

$$(f_\sharp)^* \alpha_2 = \alpha_1.$$

Proof. Given $p = (x_1, \xi_1) \in T^*M$ and $f_\sharp(x_1, \xi_1) = (x_2, \xi_2)$

$$((f_\sharp)^* \alpha_2)(p) = ((df_\sharp)^*_p(\alpha_2))(p).$$

In this way, showing that $(f_\sharp)^*\alpha_2 = \alpha_1$ is the same to show that

$$(df_\sharp)^*_p(\alpha_2(f_\sharp(p))) = \alpha_1(p).$$

Thus,

$$\begin{aligned}
(df_\sharp)^*_p(\alpha_2(f_\sharp(p))) &= (df_\sharp)^*_p(d\pi_2)^*_{f_\sharp(p)}\xi_2 \\
&= d(\pi_2 \circ f_\sharp)^*_p \xi_2 \\
&= d(f \circ \pi_1)^*_p \xi_2
\end{aligned}$$

because $\pi_2 \circ f_\sharp(x_1,\xi_1) = \pi_2(x_2,\xi_2) = x_2 = f \circ \pi_1(x_1,\xi_1)$. Then

$$\begin{aligned}(df_\sharp)_p^*(\alpha_2(f_\sharp(p))) &= (d\pi_1)_p^*(df)_p^*\xi_2\\ &= (d\pi_1)_p^*\xi_1\\ &= \alpha_p.\end{aligned}$$

□

Corollary 2.2.2. *The lift $f_\sharp$ of a diffeomorphism $f : M \to N$ is a symplectomorphism, i.e.,*

$$(f_\sharp)^*\omega_2 = \omega_1$$

where ω_1 and ω_2 are the symplectic canonical forms of M and N, respectively.

Proof. Let α_1 and α_2 being the tautological 1-forms of M and N, respectively. Thus,

$$\begin{aligned}(f_\sharp)^*\omega_2 &= (f_\sharp)^*(d\alpha_2)\\ &= d(f_\sharp)^*\alpha_2\\ &= d\alpha_1\\ &= \omega_1.\end{aligned}$$

□

Summarizing, a diffeomorphism between manifolds induces a symplectomorphism between their respective cotangent bundles, as the following diagram illustrates

$$\begin{tikzcd}(T^*M,\omega_1) \arrow[r, "f_\sharp"] & (T^*N,\omega_2)\\ M \arrow[u] \arrow[r, "f"] & N \arrow[u]\end{tikzcd}$$

As an example, take $M = N = \mathbb{S}^1(1)$. We know that $T^*\mathbb{S}^1(1)$ is the infinite cylinder $\mathbb{S}^1(1) \times \mathbb{R}$. Let $\omega = d\theta \wedge d\xi$ be the symplectic canonical form, called by **area form**. If $f : \mathbb{S}^1(1) \to \mathbb{S}^1(1)$ is a diffeomorphism, then $f_\sharp : \mathbb{S}^1(1) \times \mathbb{R} \to \mathbb{S}^1(1) \times \mathbb{R}$ is a symplectomorphism, i.e., a diffeomorphism that preserves area.

CHAPTER 3

Hamiltonian Mechanics

The aim of this chapter is to define the Hamilton formalism and study the geodesic equations in this formalism. In Section 3.1 we define the concept of Hamiltonian vector fields. In Section 3.2 we introduce the Lagrangian formalism and determine the equation of geodesics in Riemannian manifolds using such formalism. In Section 3.3 we give sufficient condition to the Hamiltonian and Lagrangian formalism to be equivalent. Finally, in Section 3.4 we find the equations of the geodesic field on the cotangent bundle. More discussion about the action of Lagrangian in the Riemannian geometry can be seen in [13]. For more details about Hamiltonian geometry and their connection with the Lagrangian formalism, see [1], [10], [37].

3.1 Hamiltonian and Symplectic Vector Fields

In this section we define the concepts of the Hamiltonian formalism and present some results. In order to make the definitions more concrete in this section, we present many examples.

Let (M,ω) be a symplectic manifold and $H : M \to \mathbb{R}$ a smooth function. Its differential is a closed 1-form dH. Since ω is non-degenerated, exist an unique vector field X_H on M such that

$$i_{X_H}\omega = dH.$$

We call X_H by **Hamiltonian vector field with Hamiltonian function** H. The triple (M,ω,H) is called by **Hamiltonian system**.

Suppose that M is compact, or at least that X_H is complete. Let $\rho_t : M \to M$ the 1-parameter group of diffeomorphisms generated by X_H. Then, ρ_t preserves the symplectic form ω. In fact,

$$\begin{aligned}\frac{d}{dt}\rho_t^*\omega &= \rho_t^*\mathscr{L}_{X_H}\omega \\ &= \rho_t^*(di_{X_H}\omega + i_{X_H}d\omega) \\ &= \rho_t^*(ddH) \\ &= 0,\end{aligned}$$

showing that $\rho_t^*\omega$ is constant for all $t \in I$. As $\rho_0 = \mathrm{Id}_M$, we have the desired. This implies that every function $H : M \to \mathbb{R}$ on (M,ω), induces a family of symplectomorphism. More than that, if (M,ω,H) is a Hamiltonian system and $\gamma(t)$ is a integral curve to X_H. By the chain rule

$$\begin{aligned}\frac{d}{dt}H(\gamma(t)) &= dH(\gamma(t))\left(\frac{d}{dt}\gamma(t)\right) \\ &= dH(\gamma(t))\,(X_H(\gamma(t))) \\ &= \omega(X_H(\gamma(t)),X_H(\gamma(t))) \\ &= 0.\end{aligned} \tag{3.1.1}$$

Therefore, $H(\gamma(t))$ is constant.

Let us see an example.

Example 3.1.1. *Let $\mathbb{S}^2(1)$ be the sphere in $\mathbb{R}^3$ with symplectic structure given by $\omega_p(u,v) = \langle p, u\times v\rangle$ for all $p \in \mathbb{S}^2(1)$ and $u,v \in T_p\mathbb{S}^2(1)$. Considering the cylindrical polar coordinates (θ,h) away from the poles, where θ is the angle and h is the height on $\mathbb{S}^2(1)$, the symplectic structure can be represented as $\omega = d\theta\wedge dh$. Consider the height function $H(\theta,h) = h$. Thus, $i_{X_H}(d\theta\wedge dh) = dh$ if and only if*

$$X_H = \frac{\partial}{\partial\theta}.$$

Then, $\rho_t(\theta,h) = (\theta+t,h)$ which is rotation about the vertical axis and the height function H is preserved by this motion.

The equalities in (3.1.1) show that the Lie derivative of H with respect to X_H is identically zero, since

$$\mathscr{L}_{X_H}H = \frac{d}{dt}H(\gamma(t))$$

with $\gamma(t)$ being the integral curve of X_H. In short, the energy functions are preserved by its Hamiltonian vector fields.

Definition 3.1.2. *A smooth vector field X on M preserving the symplectic form ω, i.e., $\mathscr{L}_X\omega = 0$, is called symplectic vector field.*

If X is a Hamiltonian vector field, exist an energy function $H : M \to \mathbb{R}$ such that X is solution of the differential equation

$$i_X\omega = dH,$$

showing that $i_X\omega$ is an exact 1-form. Conversely, if $i_X\omega$ is exact, exist $H \in \mathscr{D}(M)$ such that

$$i_X\omega = dH.$$

Therefore, the 1-form $i_X\omega$ is exact if and only if X is a Hamiltonian vector field. In particular, $i_X\omega$ is closed if X is a Hamiltonian vector field.

We have a bijection between symplectic vector fields X on M and the 1-forms $i_X\omega$ that are closed. In fact, if X is symplectic, then

$$0 = \mathscr{L}_X\omega = di_X\omega.$$

By the other side, if $i_X\omega$ is a closed 1-form, we have that

$$\mathscr{L}_X\omega = di_X\omega + i_Xd\omega = 0.$$

In summary, the following bijections are true.

$$\{X \text{ is Hamiltonian}\} \iff \{\text{The 1-form } i_X\omega \text{ is exact}\}$$

and

$$\{X \text{ is symplectic}\} \iff \{\text{The 1-form } i_X\omega \text{ is closed}\}$$

Locally, in every contractible open subset U of (M,ω), all symplectic vector fields are Hamiltonian. Indeed, since

$$H^1_{dR}(U) = 0, \tag{3.1.2}$$

if X is a symplectic vector field, by the above considerations, $i_X\omega$ is a closed 1-form. Thus, by

(3.1.2), $i_X\omega$ is exact. In the case $M = U$, we showed the existence of the following bijections

$$\{X \text{ is Hamiltonian}\} \iff \{\text{The 1-form } i_X\omega \text{ is closed}\} \iff \{X \text{ is symplectic}\}.$$

In general, $H^1_{dR}(M)$ measures the obstruction of symplectic vector fields to be Hamiltonian.

Example 3.1.3. *In the* 2*-torus* $(T^2, d\theta_1 \wedge d\theta_2)$*, the vector fields*

$$X_1 = \frac{\partial}{\partial\theta_1} \text{ and } X_2 = \frac{\partial}{\partial\theta_2}.$$

are symplectic vector fields, but they not are Hamiltonian. In fact,

$$\begin{aligned} i_{X_1}(d\theta_1 \wedge d\theta_2)(Y) &= (d\theta_1 \wedge d\theta_2)\left(\frac{\partial}{\partial\theta_1}, Y\right) \\ &= d\theta_1\left(\frac{\partial}{\partial\theta_1}\right) d\theta_2(Y) - d\theta_1(Y) d\theta_2\left(\frac{\partial}{\partial\theta_1}\right) \\ &= d\theta_2(Y). \end{aligned}$$

Analogously we have that $i_{X_2}(d\theta_1 \wedge d\theta_2)(Y) = d\theta_1(Y)$*. Therefore,* $i_{X_1}\omega$ *and* $i_{X_2}\omega$ *are closed forms, showing that they are symplectic forms. In order to show that* X_1 *and* X_2 *are not Hamiltonian, note that*

$$d\theta_1 \wedge d\theta_2 \text{ is defined in } T^2,$$

because the $d\theta_1$ *and* $d\theta_2$ *are defined in* T^2*. Considering* $x_0 \in S^1$ *and the closed curve* $\gamma(t) = (x_0, e^{2\pi i t})$ *on* T^2*, it follows that*

$$\int_{[0,1]} \gamma^* d\theta_2 = \int_{[0,1]} 2\pi dt = 2\pi,$$

concluding that $d\theta_2$ *is not exact. In analogous way, we prove that* $d\theta_1$ *is not exact.*

This example motive us to the following result.

Proposition 3.1.4. *Every Hamiltonian vector field* X *in a compact symplectic manifold* (M, ω) *is null in some point of* M*.*

Proof. Let X_H be a Hamiltonian vector field with Hamiltonian function $H : M \to \mathbb{R}$. By compactness of M, the map H has a maximum at some point $p \in M$. Thus, for any $v \in T_pM$

$$0 = dH_p(v) = i_{X_H}\omega_p(v) = \omega_p(X_H, v).$$

By non-degeneracy of ω, the Hamiltonian vector field X_H is null in $p \in M$. $\square$

3.2 Variational Principles

Let $F : TM \to \mathbb{R}$ be a smooth function. This section will be dedicated to finding conditions about F for the action to be minimized. In Subsection 3.2.1 we describe the Euler-Lagrange equations. In Subsections 3.2.2 and 3.2.3 we solve the Euler-Lagrange equations for particular cases.

3.2.1 Variational Problems

Let M be a n-dimensional smooth manifold. Let $F : TM \to \mathbb{R}$ be a smooth function.

If $\gamma : [a,b] \to M$ is a smooth curve in M, define the **lift of γ to TM** as the curve in TM given by

$$\begin{aligned} \tilde{\gamma} : [a,b] &\to TM \\ t &\mapsto (\gamma(t), \gamma'(t)). \end{aligned}$$

The **action** of γ is defined by

$$\mathscr{A}_\gamma := \int_a^b ((\tilde{\gamma}^*)F)(t)dt = \int_a^b F(\gamma(t), \gamma'(t))dt.$$

For any $p, q \in M$, denote by $\mathscr{P}(a,b,p,q)$ the set of smooth curves that start in p and ends in q, i.e.,

$$\mathscr{P}(a,b,p,q) := \{\gamma : [a,b] \to M : \gamma(a) = p \text{ and } \gamma(b) = q\}.$$

The aim of this subsection is to find among all $\gamma \in \mathscr{P}(a,b,p,q)$, the curve γ_0 which minimizes A_γ. The next lemma guarantees that minimizing curves are locally minimizing:

Lemma 3.2.1. *Suppose that $\gamma_0 : [a,b] \to M$ is minimizing. Let $[a_1,b_1]$ a subinterval of $[a,b]$ and let $p_1 = \gamma_0(a_1)$, $q_1 = \gamma_0(b_1)$. Then, $\gamma_0|_{[a_1,b_1]}$ is minimizing among the curves in $\mathscr{P}(a_1,b_1,p_1,q_1)$.*

Proof. Suppose that $\gamma_0|_{[a_1,b_1]}$ is not minimizing in $[a_1,b_1]$. Then, exist a smooth curve γ_1 connecting p_1 and q_1 that is minimizing. Define the following curve in $[a,b]$,

$$\gamma_2(t) = \begin{cases} \gamma_0(t), \text{ if } t \in [a,a_1] \\ \gamma_1(t), \text{if } t \in [a_1,b_1] \\ \gamma_0(t), \text{if } t \in [b_1,b] \end{cases} .$$

Then $\mathscr{A}_{\gamma_2} \leq \mathscr{A}_{\gamma_0}$. We can smooth the corners of the broken path $\gamma_2(t)$, in such away that we get a smooth curve $\gamma_3(t)$ satisfying $\mathscr{A}_{\gamma_3} \leq \mathscr{A}_{\gamma_0}$, which gives a contradiction. $\square$

For any $p,q \in M$ and a smooth curve γ_0 joining p and q, we assume that p,q and γ_0 are in the coordinate open subset $(x^1,\dots,x^n) : U \to \mathbb{R}^n$ of $U \subset M$. Let $(x^1,\dots,x^n,y^1,\dots,y^n) : TU \to \mathbb{R}^{2n}$ be the natural coordinate system on TU given by $y^i \frac{\partial}{\partial x^i} \mapsto (y^1,\dots,y^n)$. Using this trivialization, the curve $\gamma : [a,b] \to U$ is written in coordinates by

$$\gamma(t) = (\gamma_1(t),\dots,\gamma_n(t)).$$

The lift of γ in coordinates is

$$\tilde{\gamma}(t) = (\gamma_1(t),\dots,\gamma_n(t),\gamma_1{}'(t),\dots,\gamma_n{}'(t)).$$

Until the end of subsection we will give necessary conditions in order to a smooth curve minimizes the action. Let $c^1,\dots,c^n \in \mathscr{D}([a,b])$ such that $c^i(a) = c^i(b) = 0$ for all $1 \leq i \leq n$. Define the curve $\gamma_\varepsilon : [a,b] \to U$ by

$$\gamma_\varepsilon(t) = (\gamma_1(t) + \varepsilon c^1(t),\dots,\gamma_n(t) + \varepsilon c^n(t)).$$

For a sufficiently small $\varepsilon > 0$, the smooth curve γ_ε is well-defined and $\gamma_\varepsilon \in \mathscr{P}(a,b,p,q)$. Let $\mathscr{A}_\varepsilon := \mathscr{A}_{\gamma_\varepsilon}$. If γ_0 minimizes $\mathscr{A}$, then

$$\frac{d\mathscr{A}_\varepsilon}{d\varepsilon}(0) = 0,$$

implying that

$$
\begin{aligned}
\frac{d\mathscr{A}_\varepsilon}{d\varepsilon}(0) &= \int_a^b \frac{dF}{d\varepsilon}(\gamma_\varepsilon(t), \gamma_\varepsilon{}'(t))|_{\varepsilon=0} dt \\
&= \int_a^b \left(\frac{\partial F}{\partial x^i}(\gamma_0(t), \gamma_0{}'(t)) c^i(t) + \frac{\partial F}{\partial y^i}(\gamma_0(t), \gamma_0{}'(t)) c^{i\prime}(t) \right) \\
&= \int_a^b \frac{\partial F}{\partial x^i}(\gamma_0(t), \gamma_0{}'(t)) c^i(t) + \frac{\partial F}{\partial y^i}(\gamma_0(t), \gamma_0{}'(t)) c^i(t) \Big|_a^b - \int_a^b \frac{d}{dt}\frac{\partial F}{\partial y^i}(\gamma_0(t), \gamma_0{}'(t)) c^i(t) \\
&= \int_a^b \left(\frac{\partial F}{\partial x^i}(\gamma_0(t), \gamma_0{}'(t)) - \frac{d}{dt}\frac{\partial F}{\partial y^i}(\gamma_0(t), \gamma_0{}'(t)) \right) c^i(t) dt \\
&= 0.
\end{aligned}
$$

Since this is true for all $c^i(t), 1 \leq i \leq n$, satisfying the boundary condition $c^i(a) = c^i(b) = 0$, by the Fundamental Lemma of the Calculus of Variations (see page 6 of [24]), we conclude that

$$
\frac{\partial F}{\partial x^i}(\gamma_0(t), \gamma_0{}'(t)) = \frac{d}{dt}\frac{\partial F}{\partial v^i}(\gamma_0(t), \gamma_0{}'(t)). \tag{3.2.1}
$$

These are the **Euler-Lagrange equations** of the problem of minimizing the action $\mathscr{A}_\gamma$ with $\gamma \in \mathscr{P}(a, b, p, q)$.

3.2.2 Minimizing Properties

From now on, assume that

$$
\det\left(\frac{\partial^2 F}{\partial v^i \partial v^j} \right) \neq 0. \tag{3.2.2}
$$

The expression (3.2.2) is called **Legendre condition**. Taking

$$
g^{ij}(p, v) = \left(\frac{\partial^2 F}{\partial v^i \partial v^j} \right)^{-1},
$$

the Euler-Lagrange equations become

$$
(\gamma^j)''(t) = g^{ij}\frac{\partial F}{\partial x^i}(\gamma(t), \gamma'(t)) - g^{ij}\frac{\partial^2 F}{\partial v^i \partial x^k}(\gamma(t), \gamma'(t))(\gamma^k)'(t). \tag{3.2.3}
$$

This second order ordinary differential equation has a unique solution given initial conditions

$$
\gamma(a) = p \text{ and } \gamma'(t) = v.
$$

In this subsection we study minimizing properties of the solution (3.2.3).

Let $\mathbb{V}$ a n-dimensional real vector space with basis $\{e_1,\dots,e_n\}$. Given $v\in\mathbb{V}$, we write $v=v^ie_i$. Let $F(v^1,\dots,v^n):\mathbb{V}\to\mathbb{R}$ be a smooth map on $\mathbb{V}$. Take $p\in\mathbb{V}$ and $u=u^ie_i\in\mathbb{V}$. The **Hessian of** F is the quadratic map on $\mathbb{V}$ defined by

$$(d^2F)_p(u):=\frac{\partial^2F(p)}{\partial v^i\partial v^j}u^iu^j.$$

Definition 3.2.2. *We say that a function $F:\mathbb{V}\to\mathbb{R}$ is strongly convex if $(d^2F)_p$ is positive-definite for all $p\in\mathbb{V}$.*

An important question is if $\gamma_0\in\mathscr{P}(a,b,p,q)$ satisfies the Euler-Lagrange equations, does γ_0 minimizes $\mathscr{A}$? According to the next theorem, locally the answer is yes.

Theorem 3.2.3. *For every sufficiently small subinterval $[a_1,b_1]$ of $[a,b]$, the curve $\gamma_0|_{[a_1,b_1]}$ is minimizing in $\mathscr{P}(a_1,b_1,p_1,q_1)$.*

The proof of this theorem needs the Wirtinger inequality, which will be assumed.

Lemma 3.2.4 (Wirtinger inequality)**.** *Let $f:[a,b]\to\mathbb{R}$ be a C^1-class map with $f(a)=f(b)=0$. Then,*

$$\int_a^b|f(t)|^2dt\le\frac{(b-a)^2}{\pi^2}\int_a^b|f'(t)|^2dt.$$

Proof. For a proof see page 184 of [21]. □

The Wirtinger inequality has many applications in geometry. As an example of application, see the isoperimetric inequality in [21]. Now, we use the Wirtinger inequality to prove Theorem 3.2.3.

Suppose that $\gamma_0:[a,b]\to U$ satisfies the Euler-Lagrange equation. Take $c^i\in\mathscr{D}([a,b])$ such that $c^i(a)=c^i(b)=0$ for $1\le i\le n$. Define $\gamma_\varepsilon=\gamma_0+\varepsilon c$ where $c=(c^1,\dots,c^n)$. For sufficiently small $\varepsilon>0$, the smooth curve $\gamma_\varepsilon\in\mathscr{P}(a,b,p,q)$ and define $\mathscr{A}_\varepsilon=\mathscr{A}_{\gamma_\varepsilon}$. We have that

$$\frac{d^2\mathscr{A}_\varepsilon(0)}{d\varepsilon}=\int_a^b\frac{\partial^2F}{\partial x^i\partial x^j}(\gamma_0(t),\gamma_0'(t))c^i(t)c^j(t)dt \tag{3.2.4}$$

$$+2\int_a^b\frac{\partial^2F}{\partial x^i\partial v^j}(\gamma_0(t),\gamma_0'(t))c^i(t)c^{j\prime}(t)dt \tag{3.2.5}$$

$$+\int_a^b\frac{\partial^2F}{\partial v^i\partial v^j}(\gamma_0(t),\gamma_0'(t))c^{i\prime}(t)c^{j\prime}(t)dt. \tag{3.2.6}$$

Analyzing the term (3.2.4),

$$\begin{aligned}\left|\int_a^b \frac{\partial^2 F}{\partial x^i \partial x^j}(\gamma_0(t),\gamma_0{}'(t))c_i(t)c_j(t)dt\right| &\leq \int_a^b |\langle c(t),(d^2F)_p(c(t))\rangle|dt\\ &\leq K_1|c|^2_{L_2},\ \text{for some constant } K_1>0\end{aligned}$$

where $|\cdot|_{L_2}$ is the L_2 norm. By the same argument and Hölder inequality (see page 92 of [7]) in (3.2.5) we have that:

$$\begin{aligned}\left|2\int_a^b \frac{\partial^2 F}{\partial x^i \partial v^j}(\gamma_0(t),\gamma_0{}'(t))c^i(t)c^{j\prime}(t)dt\right| &\leq K_2\left|\int_a^b c^i(t)c^{j\prime}(t)dt\right|\\ &\leq K_2\int_a^b |\langle c(t),c\,'(t)\rangle|dt\\ &\leq K_2|c|_{L_2}|c'|_{L_2}.\end{aligned}$$

for some constant $K_2>0$ bounding $\frac{\partial^2 F}{\partial x^i \partial v^i}(\gamma_0(t),\gamma_0{}'(t))$. To get a constant in term (3.2.6), as d^2F is definite-positive for all $(p,v)\in TM$, there exists a constant $K_3>0$ such that

$$\int_a^b \frac{\partial^2 F}{\partial v^i \partial v^j}(\gamma_0(t),\gamma_0{}'(t))c_i{}'(t)c_j{}'(t)dt \geq K_3|c\,'|^2_{L_2}.$$

Therefore,

$$\begin{aligned}\frac{d^2\mathscr{A}_\varepsilon}{d\varepsilon^2}(0) &\geq K_3|c\,'|^2_{L_2}-2K_2|c|_{L_2}|c\,'|_{L_2}-K_1|c|^2_{L_2}\\ &\geq \left(K_3-\frac{K_3}{2}\right)\int_a^b|c\,'(t)|^2-\left(\frac{K_2^2}{2K_3}+K_1\right)\int_a^b|c(t)|^2\end{aligned}$$

were we use that

$$\frac{K_2^2}{2K_3}A^2+\frac{K_3}{2}B^2\geq K_2AB$$

for terms A,B. Putting $K_4:=\frac{K_3}{2}$ and $K_5:=\frac{K_2^2}{2K_3}+K_1$, by Wirtinger inequality

$$\begin{aligned}\frac{d^2A_\varepsilon}{d\varepsilon^2}(0) &\geq \frac{K_4\pi^2}{(b-a)^2}|c|^2_{L_2}-K_5|c|^2_{L_2}\\ &\geq \left(K_4\frac{\pi^2}{(b-a)^2}-K_5\right)|c|^2_{L_2}.\end{aligned}$$

For sufficiently small $b-a$, we showed that

$$\frac{d^2\mathscr{A}_\varepsilon}{d\varepsilon^2}(0) > 0.$$

As γ_0 satisfies Euler-Lagrange, then γ_0 is locally minimizing.

3.2.3 Minimizing Geodesics

In this subsection we restrict the study for locally minimizing curves in a smooth Riemannian manifold. Let $(M, \langle \cdot, \cdot \rangle)$ be a n-dimensional Riemannian manifold. The Riemannian metric induces a quadratic map

$$\begin{aligned} F : TM &\to \mathbb{R} \\ (p,v) &\mapsto |v|_p^2 \end{aligned}$$

where $|v|_p^2 = \langle v, v \rangle_p$. Let $\gamma\colon [a,b] \to M$ be a smooth curve joining p and q on M. By the quadratic map F the action of γ is given by

$$\mathscr{A}_\gamma = \int_a^b |\gamma\,'(t)|^2 dt.$$

Let $\tau : [a_1, b_1] \to [a,b]$ be a reparametrization of γ. Note that the arc-length of a smooth curve is invariant under reparametrizations. In fact, if τ is increasing, then

$$\begin{aligned} l(\gamma \circ \tau) &= \int_{a_1}^{b_1} |(\gamma \circ \tau)\,'(t)| dt \\ &= \int_{a_1}^{b_1} |\gamma\,'(\tau(t))\tau\,'(t)| dt \\ &= \int_{a_1}^{b_1} \tau'(t)|\gamma\,'(\tau(t))| dt \\ &= \int_a^b |\gamma\,'(u)| du \\ &= l(\gamma) \end{aligned}$$

where $u(t) = \tau(t)$ and $du = \tau\,'(t)dt$. The proof when τ is decreasing is analogous. Supposing that $\gamma\,'(t) \neq 0$ for all $t \in [a,b]$, we can find a reparametrization $\tau : [a,b] \to [a,b]$ such that $\gamma \circ \tau : [a,b] \to M$ has constant velocity. We say that a smooth curve $\gamma\colon [a,b] \to M$ parametrized in this way is parametrized proportionally to the arc-length.

Let $\gamma : [a,b] \to M$ be a smooth curve on M. By the Schwarz inequality:

$$\left(\int_a^b 1 \cdot |\gamma'(t)| dt\right)^2 \leq \left(\int_a^b 1^2 dt\right)\left(\int_a^b |\gamma'(t)|^2 dt\right).$$

Thus,

$$l(\gamma)^2 \leq (b-a)\mathscr{A}_\gamma$$

and equality occurs if and only if $|\gamma'(t)|$ is constant, that is, if and only if t is proportional to the arc-length. These computations take us to the following proposition:

Proposition 3.2.5. *Let $p,q \in M$ and let $\gamma_0 : [a,b] \to M$ be a minimizing geodesic joining p and q. Then, for all curves $\gamma \in \mathscr{P}(a,b,p,q)$*

$$\mathscr{A}_{\gamma_0} \leq \mathscr{A}_\gamma$$

and the equality holds if and only if γ is a minimizing geodesic.

Proof. By the computations above, it follows that

$$\begin{aligned}(b-a)\mathscr{A}_{\gamma_0} &= l(\gamma_0)^2 \\ &\leq l(\gamma)^2 \\ &\leq (b-a)\mathscr{A}_\gamma\end{aligned}$$

for all $\gamma \in \mathscr{P}(a,b,p,q)$, proving the first part. Now, if the equality holds, we have that $l(\gamma)^2 = (b-a)\mathscr{A}_\gamma$, implying that the parameter t is proportional to arc-length, and $l(\gamma_0) = l(\gamma)$. The converse is trivial by the first part of the proof. □

Let $(x^1,\ldots,x^n) : U \to \mathbb{R}^n$ be a coordinate system on the open set $U \subset M$ with associated chart in the tangent bundle TM given by $(x^1,\ldots,x^n,y^1,\ldots,y^n) : TU \to \mathbb{R}^{2n}$. In local coordinates the quadratic map is given by

$$F(p,v) = g_{ij}(p)v^i v^j$$

where $g_{ij}(p) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_p$. We shall to compute the Euler-Lagrange equations for this quadratic map. By the left hand side of (3.2.1),

$$\begin{aligned}\frac{\partial F}{\partial x^j}(\gamma,\dot\gamma(t)) &= \frac{\partial}{\partial x^j}(g_{ij}(p)\dot\gamma^i\dot\gamma^j)\\ &= \frac{\partial}{\partial x^j}g_{ik}(p)\dot\gamma^i\dot\gamma^j.\end{aligned}$$

By the right side of (3.2.1),

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial F}{\partial v^j}(\gamma,\dot\gamma(t))\right) &= \frac{d}{dt}(g_{ik}(p)\dot\gamma^i)\\ &= \frac{\partial g_{ik}(p)}{\partial x^j}\dot\gamma^i\dot\gamma^j + g_{ik}(p)\ddot\gamma^i\\ &= \frac{1}{2}\frac{\partial g_{ik}(p)}{\partial x^j}\dot\gamma^i\dot\gamma^j + \frac{1}{2}\frac{\partial g_{ik}(p)}{\partial x^j}\dot\gamma^i\dot\gamma^j + g_{ik}(p)\ddot\gamma^k.\end{aligned}$$

Interchanging the indices i and j in the middle term, we have that

$$\frac{d}{dt}\left(\frac{\partial F}{\partial v^j}(\gamma,\dot\gamma(t))\right) = \frac{1}{2}\frac{\partial g_{ik}(p)}{\partial x^j}\dot\gamma^i\dot\gamma^j + \frac{1}{2}\frac{\partial g_{jk}(p)}{\partial x^i}\dot\gamma^i\dot\gamma^j + g_{ik}(p)\ddot\gamma^k.$$

Joining the two sides, the Euler-Lagrange equations are

$$g_{ik}\ddot\gamma^i + \frac{1}{2}\left(\frac{\partial g_{ik}(p)}{\partial x^j} + \frac{\partial g_{jk}(p)}{\partial x^i} - \frac{\partial g_{ij}(p)}{\partial x^k}\right)\dot\gamma^i\dot\gamma^j = 0.$$

Multiplying by the inverse matrix g^{ik} of g_{ik}, it follows that

$$\ddot\gamma^i + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0 \tag{3.2.7}$$

where

$$\Gamma^s_{ij}(p) = \frac{1}{2}\left(\frac{\partial g_{ik}(p)}{\partial x^j} + \frac{\partial g_{jk}(p)}{\partial x^i} - \frac{\partial g_{ij}(p)}{\partial x^k}\right)g^{ks}.$$

Definition 3.2.6. *A smooth curve $\gamma : [a,b] \to M$ that locally minimizes the arc-length is called* ***geodesic****.*

The second order differential equation (3.2.7) is called **geodesic equation**

3.3 Legendre Transform

In this section we finally give the connection between the Lagrangian formalism and the Hamiltonian formalism. The Subsection 3.3.1 is preliminary for the Subsection 3.3.2, where we study the concept of Legendre transform.

3.3.1 Strong Convexity

In the literature, the maps satisfying the conditions in Definition 3.2.2 are usually called strictly convex maps, but these maps are usually related to a weaker definition. Let $\mathbb{V}$ be an n-dimensional real vector space. In this dissertation a map $f : \mathbb{V} \to \mathbb{R}$ will be called convex if for any $v, w \in \mathbb{V}$

$$f(tv + (1-t)w) \leq tf(v) + (1-t)f(w), \text{ for all } t \in [0,1].$$

The map f is called strictly convex if the inequality above is strict for all $t \in (0,1)$. In this section we explore the relation between strictly and strongly convex maps and explore their properties.

For any $v, w \in \mathbb{V}$, denote

$$(v,w) = \{z \in \mathbb{V} : z = tv + (1-t)w \text{ where } t \in (0,1)\}$$

and

$$[v,w] = \{z \in \mathbb{V} : z = tv + (1-t)w \text{ where } t \in [0,1]\}$$

the open and closed segment joining v and w, respectively.

Let $f : \mathbb{V} \to \mathbb{R}$ be a strongly convex map. For any $v, w \in \mathbb{V}$, the Taylor formula with Lagrange remainder guarantees the existence of a element $z \in (v,w)$ such that

$$f(w) = f(v) + (df)_v(w-v) + (d^2f)_z(w-v)^2.$$

By Definition 3.2.2,

$$f(w) > f(v) + \frac{1}{2}(df)_v(w-v).$$

Therefore, if $x = t_0 v + (1-t_0)w$, we have that

$$f(v) > f(x) + (df)_x(v-x) \tag{3.3.1}$$

$$f(w) > f(x) + (df)_x(w-x). \tag{3.3.2}$$

Multiplying (3.3.1) by t_0, (3.3.2) by $(1-t_0)$ and adding the two inequalities, it follows that

$$\begin{aligned}
t_0 f(v) + (1-t_0)f(w) &> t_0 f(x) + t_0(df)_x(v-x) + f(x) - t_0 f(x) \\
&\quad + (df)_x(w-x) - t_0(df)_x(w-x) \\
&= (df)_x(t_0 v - t_0 x + w - x - t_0 w + t_0 x) + f(x) \\
&= f(x).
\end{aligned}$$

This implies that f is a strictly convex map.

Remark 3.3.1. *A strictly convex map $f : \mathbb{V} \to \mathbb{R}$ is not necessarily strongly convex. In fact, the map $f(x) = x^4$ is strictly convex, but $f''(0) = 0$.*

Remark 3.3.2. *A map $f : \mathbb{V} \to \mathbb{R}$ is strongly convex if and only if it is strongly convex when restricted to lines. This means that the map $f : \mathbb{V} \to \mathbb{R}$ is strongly convex if and only if $f(v+tw)$ is strongly convex for every $v \in \mathbb{V}$ and $w \in \mathbb{V} \setminus \{0\}$. This fact follows when we define $g(t) = f(v+tw)$. Thus,*

$$g''(t) = (d^2 f)_{v+tw}(w)^2.$$

Therefore, $g''(t) > 0$ if and only if $(d^2 f)_{v+tw}(w)^2$ for all $v, w \in \mathbb{V}$ with $v \neq w$.

The following proposition give us many conditions which are equivalent to strong convexity of a map.

Proposition 3.3.3. *Let $f : \mathbb{V} \to \mathbb{R}$ be a strongly convex map. The following items are equivalent:*

1. *f has a critical point;*
2. *f has a local minimum at some point;*
3. *f has a unique global minimum;*
4. *f is proper, i.e., $f(v) \to +\infty$ when $|v| \to +\infty$.*

Proof. (1) $\Rightarrow$ (2) If $(df)_v = 0$ for some $v \in \mathbb{V}$, by the Taylor formula with Lagrange remainder, for any $w \in \mathbb{V}$, exist $x \in \mathbb{V}$ such that

$$\begin{aligned} f(w) &= f(v) + (df)_v(w-v) + \frac{1}{2}(d^2f)_x(w-v)^2 \\ &> f(v). \end{aligned}$$

for $w \neq v$ because f is strongly convex. Thus $f(v)$ is the unique global minimum of f. In particular, is a local minimum.

(2) $\Rightarrow$ (3) The prove of this implication just proceed as in (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (1) If $f(v)$ is a unique global minimum for some $v \in \mathbb{V}$, then $(df)_v = 0$.

(1) $\Rightarrow$ (4) The condition $f(v) \to +\infty$ when $|v| \to +\infty$ means that for any constant $M \in \mathbb{R}$, there exist $R_M > 0$ such that $f(v) \geq M$ whenever $|v| \geq R_M$. Suppose without loss of generality that $f(0) = 0$ is the unique critical point of f. Let $\mathbb{S}^{n-1}(1) \subset \mathbb{V}$ the unit sphere on $\mathbb{V}$. Since $\mathbb{S}^{n-1}(1)$ is compact, exist $v_1 \in \mathbb{S}^{n-1}(1)$ such that $f|_{\mathbb{S}^{n-1}(1)}(v_1)$ is the minimum. Define

$$\begin{aligned} \varphi : \mathbb{V} \setminus \{0\} &\to \mathbb{R} \\ v &\mapsto \frac{f(v)}{|v|}. \end{aligned}$$

We have that φ is the slope of the line passing through 0 and $f(v)$. φ is well-defined and smooth. As $f|_{\mathbb{S}^{n-1}(1)}(v_1)$ is the minimum on $\mathbb{S}^{n-1}(1)$, it follows that

$$\lambda_{\min} := \varphi(v_1) \leq \varphi(w), \text{ for every } w \in \mathbb{S}^{n-1}(1).$$

Fix $x \in \mathbb{S}^{n-1}(1)$ and consider the ray σ_x starting in 0 and passing through x. Strong convexity of f guarantees that for any $y \in \sigma_x$ with $|y| \geq 1$,

$$\varphi(x) = \frac{f(x)}{|x|} \leq \frac{f(y)}{|y|} = \varphi(y).$$

Then, given $M > 0$, define $R_M := \frac{M}{\lambda_{\min}}$. If $|y| > R_M$, this implies that

$$f(y) \geq \lambda_{\min}|y| > \lambda_{\min}\left(\frac{M}{\lambda_{\min}}\right) = M.$$

(4) $\Rightarrow$ (1) Let $M > 0$ such that $(-\infty, M] \cap \mathrm{Im} f \neq \emptyset$. Let $R_M > 0$ such that if $|y| > R_M$, then

$f(y) > M$. Then $f^{-1}((-\infty, M]) \subset B[0, R_M]$ and it is a nonempty compact subset of $\mathbb{V}$. Therefore f assumes a global minimum in $f^{-1}((-\infty, M])$, which is also a minimum in $\mathbb{V}$. $\square$

Definition 3.3.4. *A strongly convex map $f : \mathbb{V} \to \mathbb{R}$ is said to be* ***stable*** *if satisfies any condition of Proposition 3.3.3.*

Next let us present some examples:

Example 3.3.5. *Consider the map $f(x) = e^x + ax$. As $f''(x) = e^x > 0$ for all $x \in \mathbb{R}$, we have that f is strongly convex. $f'(x) = e^x + a$, and f' admits a critical point if and only if $a < 0$. In this case, f is stable. It isn't stable for $a \geq 0$.*

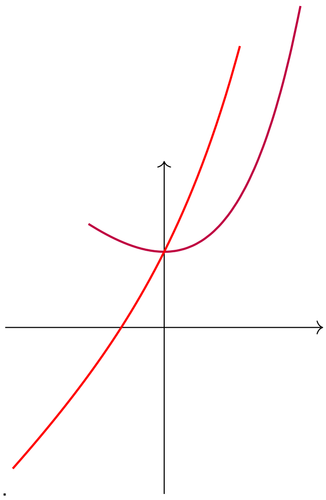

Figure 3.1: Function $e^x + ax$ for $a = -1$ and $a = 1$ in purple and red, respectively.

Example 3.3.6. *The map $g(x) = x^2 + ax$ is strongly convex, because $g''(x) = 2 > 0$ for all $x \in \mathbb{R}$. It is straightforward that g is stable for all $a \in \mathbb{R}$. Some solutions are presented below:*

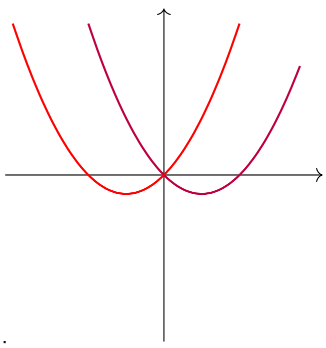

Figure 3.2: Function $x^2 + ax$ for $a = -1$ and $a = 1$ in purple and red, respectively.

3.3.2 Legendre Transform

For now on, we will assume that $f : \mathbb{V} \to \mathbb{R}$ is strongly convex. For each $v \in \mathbb{V}$ we have the identification

$$T_v^*\mathbb{V} = \mathbb{V}^*.$$

By this identification we can consider the section df on the cotangent bundle as a map

$$df : \mathbb{V} \to \mathbb{V} \times \mathbb{V}^*.$$

Projecting on the second factor, the Legendre transform is defined by

$$\begin{aligned} L_f : \mathbb{V} &\to \mathbb{V}^* \\ v &\mapsto (df)_v. \end{aligned}$$

If $(v^1, \ldots, v^n)$ is a coordinate system on $\mathbb{V}$, we have that

$$\det(d^2 f)_v > 0, \text{ for all } v \in \mathbb{V}.$$

This implies that the Legendre transform is a local diffeomorphism by the inverse mapping theorem. Below, we will show that L_f takes $\mathbb{V}$ diffeomorphically on an open subset of $\mathbb{V}^*$. Given $\alpha \in \mathbb{V}^*$, define

$$\begin{aligned} f_\alpha : \mathbb{V} &\to \mathbb{R} \\ v &\mapsto f(v) - \alpha(v). \end{aligned}$$

Observe that

$$\begin{aligned} (d^2 f_\alpha)_v(u) &= \frac{d^2 f_\alpha}{dt^2}(v+tu)|_{t=0} \\ &= \frac{d^2(f-\alpha)}{dt^2}(v+tu)|_{t=0} \\ &= (d^2 f)_v(u). \end{aligned}$$

Definition 3.3.7. *The set of stability of a strongly convex map $f : \mathbb{V} \to \mathbb{R}$ is defined by*

$$W_f := \{\alpha \in \mathbb{V}^* : f_\alpha \textit{ is stable}\}.$$

Proposition 3.3.8. *Let $f : \mathbb{V} \to \mathbb{R}$ be a strongly convex map. The set W_f is open and convex in $\mathbb{V}^*$. Moreover the Legendre transform L_f is a diffeomorphism over W_f.*

Proof. First of all, notice that $df_v \in W_f$ for every $v \in \mathbb{V}$ because f_{df_v} admits a critical point. Therefore $L_f(\mathbb{V}) \subset W_f$.

Conversely, take $\alpha \in W_f$. Since $f_\alpha : \mathbb{V} \to \mathbb{R}$ is stable, exist an unique $v \in \mathbb{V}$ such that $(df_\alpha)_v = 0$. Therefore,

$$\begin{aligned} 0 &= (df_\alpha)_v(u) \\ &= (df)_v(u) - \alpha(u), \end{aligned}$$

i. e., $df_v = \alpha$, and $W_f \subset L_f(\mathbb{V})$. In short, this shows that for any $\alpha \in W_f$, there exists a unique $v \in \mathbb{V}$ such that $df_v = \alpha$, showing that the Legendre transform L_f is a bijection over W_f. As L_f is a local diffeomorphism for every $v \in \mathbb{V}$, there exist neighborhoods U_v of v on $\mathbb{V}$ and U_α of α on $\mathbb{V}^*$ such that $L_f|_{U_v}$ is a diffeomorphism over U_α. Therefore, $U_\alpha \subset W_f$ is a neighborhood of α contained in W_f, showing that W_f is a open set. The fact that L_f is a bijection implies that L_f is a diffeomorphism over W_f due to the inverse function theorem.

Take $\alpha_1, \alpha_2 \in W_f$. Since f_{α_1} and f_{α_2} are stable functions, we have that

$$f_{\alpha_1}(v), f_{\alpha_2}(v) \to +\infty, \text{ when } |v| \to +\infty. \tag{3.3.3}$$

Now, for any $t \in [0,1]$,

$$t f_{\alpha_1}(v) + (1-t) f_{\alpha_2}(v) = f_{t\alpha_1 + (1-t)\alpha_2}(v). \tag{3.3.4}$$

From (3.3.3) and (3.3.4), it follows that $f_{t\alpha_1+(1-t)\alpha_2}(v) \to +\infty$ when $|v| \to +\infty$ and $t\alpha_1 + (1-t)\alpha_2 \in W_f$, what shows that W_f is convex. □

As consequence of Legendre transform being a bijection, we have a global minimum of f_α for each $\alpha \in W_f$. Indeed, given $\alpha \in W_f$, there exists a unique point $v \in \mathbb{V}$ such that $v = L_f^{-1}(\alpha)$ and $L_f(v)(u) = \alpha(u)$ for every $u \in \mathbb{V}$. Thus, $(df_\alpha)_v(u) = 0$. Since, f_α is stable, we have that $f_\alpha(v)$ is the global minimum of f_α on $\mathbb{V}$.

Definition 3.3.9. *The **dual map** f^* of f is defined by*

$$\begin{aligned} f^* : W_f &\to \mathbb{R} \\ \alpha &\mapsto -\min_{v \in \mathbb{V}} f_\alpha(v). \end{aligned} \tag{3.3.5}$$

We have that,

$$f^*(\alpha) = \alpha(v) - f(v),$$

where $v = L_f^{-1}(\alpha)$, due to Proposition 3.3.8.

Theorem 3.3.10. *Assuming that $W_f = \mathbb{V}^*$, we have that $L_f^{-1} = L_{f^*}$.*

Proof. First of all notice that

$$f^*(\alpha) = \alpha(L_f^{-1}(\alpha)) - f(L_f^{-1}(\alpha)).$$

Denote $v = L_f^{-1}(\alpha)$. Considering the derivative of f^* at α calculated at β, we get

$$df^*_\alpha(\beta) = \beta(L_f^{-1}(\alpha)) + \alpha(d(L_f^{-1})_\alpha(\beta)) - df_v(d(L_f^{-1})_\alpha(\beta)).$$

But $\alpha = df_v$. Therefore

$$df^*_\alpha(\beta) = \beta(L_f^{-1}(\alpha)).$$

Identifying $df^*_\alpha \in \mathbb{V}^{**}$ with $L_f^{-1}(\alpha) = (df)_\alpha^{-1} \in \mathbb{V}$, we get $(L_f)^{-1} = L_{f^*}$. □

3.4 Variational Problems

The goal of this section is to give the relation between the Hamiltonian and Lagrangian formalism. In Subsection 3.4.1 this relation is established. In Subsection 3.4.2 we use this relation to study geodesics. As a corollary, we gain a geodesic equation in the Hamiltonian formalism.

3.4.1 Application to Variational Problems

Let M be a n-dimensional smooth manifold and $F : TM \to \mathbb{R}$ be a smooth function on the tangent bundle TM. The goal of this section is to study the problem of minimization of $\mathscr{A}_\gamma$. For any $p \in M$, denote

$$F_p := F|_{T_pM} : T_pM \to \mathbb{R}.$$

Assume that F_p is strongly convex for any $p \in M$. The Legendre transform in each tangent space

$$L_{F_p} : T_pM \to W_{F_p}$$

is a diffeomorphism. The dual map to F_p is denoted by $F_p^* : W_{F_p} \to \mathbb{R}$. With all this notations in place and assuming that $W_{F_p} = T_p^*M$ for every $p \in M$, we can define

$$\begin{aligned} L &: TM \to T^*M \\ &(p,v) \to L(p,v) := d(F_p)_v : T_pM \to \mathbb{R} \end{aligned}$$

and

$$\begin{aligned} H &: T^*M \to \mathbb{R} \\ &(p,\alpha) \mapsto H(p,\alpha) = F_p^*(\alpha) : T_p^*M \to \mathbb{R}. \end{aligned}$$

The maps L and H are smooth maps on their respective vector bundles. In fact, the map L is a diffeomorphism, because the restriction of L to each tangent space is also a diffeomorphism.

Lemma 3.4.1. *Let $(x,\xi) \in \Gamma_L$, where Γ_L is the graph of L. Then,*

$$\frac{\partial H}{\partial x}(x,\xi) = -\frac{\partial F}{\partial x}(x,v)$$

with $H = F^*$.

Proof. First, note that

$$\begin{aligned} \frac{\partial H}{\partial x}(x,\xi) &= L_{F_x^*}(x,\xi) \\ &= L_{F_x^*}(L_{F_x}(v)) \\ &= (x,v) \end{aligned}$$

where $v = v(x,\xi)$ is the unique $v \in T_xM$ satisfying $(x,\xi) = L(x,v(x,\xi))$. Deriving $H(x,\xi) =$

$\xi(v(x,\xi)) - F(x, v(x,\xi))$ with respect to x, we have

$$\frac{\partial H}{\partial x} = \xi\left(\frac{\partial v}{\partial x}\right) - \frac{\partial F}{\partial x}(x,v) - \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}\right).$$

But $\xi = \frac{\partial F}{\partial v}$ because ξ is the Legendre transform of F and

$$\frac{\partial H}{\partial x} = -\frac{\partial F}{\partial x}$$

what settles the lemma. □

Let $(x^1,\ldots,x^n) : U \to \mathbb{R}^n$ be a local coordinate systems on the open set $U \subset M$ with associate coordinate systems $(x^1,\ldots,x^n,v^1,\ldots,v^n) : TU \to \mathbb{R}^{2n}$ and $(x^1,\ldots,x^n,\xi_1,\ldots,\xi_n) : T^*U \to \mathbb{R}^{2n}$ for TM and T^*M, respectively. Let $\gamma : [a,b] \to M$ a smooth curve and $\tilde{\gamma} : [a,b] \to TM$ its lift. In local coordinates, write $\gamma(t) = (x^1(t),\ldots,x^n(t))$ and $\tilde{\gamma}(t) = (x^1(t),\ldots,x^n(t),v^1(t),\ldots,v^n(t))$.

Theorem 3.4.2. *The smooth curve γ satisfies the Euler-Lagrange equation if and only if $L \circ \tilde{\gamma}$: $[a,b] \to T^*M$ is a integral curve of the Hamiltonian field X_H where $H(p,\alpha) = F_p^*(\alpha)$.*

Proof. We know that the integral curves $(x(t),\xi(t))$ of X_H satisfies the Hamilton equations:

$$\begin{cases} \frac{dx}{dt}(t) = \frac{\partial H}{\partial \xi}(x(t),\xi(t)); \\ \frac{d\xi}{dt}(t) = -\frac{\partial H}{\partial x}(x(t),\xi(t)). \end{cases}$$

$x(t)$ satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial x}(x(t),x'(t)) = \frac{d}{dt}\frac{\partial F}{\partial v}(x(t),x'(t)).$$

Define $v(t) = x'(t)$ and $L(x(t),\xi(t))$. Suppose that γ satisfies the Euler-Lagrange equations. As $L_{F_{x(t)}}(v(t)) = \xi(t)$, then

$$\frac{dx}{dt}(t) = L_{F^*_{x(t)}}(\xi(t)) = \frac{\partial H}{\partial \xi}(x(t),\xi(t))$$

satisfying the first condition of Hamilton equations. Now, by $\xi = \frac{\partial F}{\partial v}$ and Lemma 3.4.1, we have

$$\frac{d}{dt}\frac{\partial F}{\partial v}(x(t),v(t)) = \frac{d\xi}{dt}(t) \tag{3.4.1}$$

and

$$\frac{\partial F}{\partial x}(x(t),v(t)) = -\frac{\partial H}{\partial x}(x(t),\xi(t)). \tag{3.4.2}$$

Joining the equations (3.4.1) and (3.4.2), it follows that

$$\frac{d\xi}{dt}(t) = -\frac{\partial H}{\partial x}(x(t),\xi(t)),$$

showing that $(x(t),\xi(t))$ satisfies the Hamilton equation. By the other side, if $(x(t),\xi(t))$ satisfies the Hamilton equation,

$$\frac{d}{dt}\frac{\partial F}{\partial v}(x(t),v(t)) = \frac{d\xi}{dt}(t) = -\frac{\partial H}{\partial x}(x(t),\xi(t)) = \frac{\partial F}{\partial x}(x(t),v(t)),$$

proving the desired equivalence. □

3.4.2 Geodesics Equation in Hamiltonian Formalism

Let $(M,\langle\cdot,\cdot\rangle)$ be a n-dimensional Riemannian manifold with Riemannian metric $\langle\cdot,\cdot\rangle$. Let $(x^1,\ldots,x^n): U \to \mathbb{R}^n$ be a local coordinate system on the open set $U \subset M$ with corresponding natural coordinate systems $(x^1,\ldots,x^n,y^1,\ldots,y^n): TU \to \mathbb{R}^{2n}$ and $(x^1,\ldots,x^n,\xi_1,\ldots,\xi_n): T^*U \to \mathbb{R}^{2n}$. The map

$$\begin{aligned} F : TM &\to \mathbb{R} \\ (p,v) &\mapsto \frac{1}{2}g_{ij}(p)v^iv^j \end{aligned}$$

is strongly convex where $g_{ij} = \langle\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\rangle$. Let $\gamma:[a,b]\to M$ be a geodesic joining p and q on M and $\tilde{\gamma}:[a,b]\to M$ its lift. In local coordinates, write $\gamma(t) = (x^1(t),\ldots,x^n(t))$ and $\tilde{\gamma}(t) = (x^1(t),\ldots,x^n(t),v^1(t),\ldots,v^n(t))$. Then,

$$\begin{aligned} \xi_k(t) &= \frac{\partial F_{x(t)}}{\partial v^k}(\dot{x}(t)) \\ &= \frac{\partial}{\partial v^k}\left(\frac{1}{2}g_{ij}(x(t))\dot{x}^i\dot{x}^j\right) \\ &= \frac{1}{2}g_{ij}(x(t))\delta^i_k\dot{x}^j + \frac{1}{2}g_{ij}(x(t))\delta^j_k\dot{x}^i \\ &= \frac{1}{2}g_{kj}(x(t))\dot{x}^j + \frac{1}{2}g_{ik}(x(t))\dot{x}^i \end{aligned}$$

$$= g_{ki}(x(t))\dot{x}^i$$

Therefore, $\dot{x}^i(t) = g^{ki}(x(t))\xi_k(t)$. But,

$$\begin{aligned}
H((x(t),\xi(t)) &= \xi_i\dot{x}^i - \frac{1}{2}g_{ij}(x(t))\dot{x}^i\dot{x}^j \\
&= \xi_i g^{ki}(x(t))\xi_k - \frac{1}{2}g_{ij}(x(t))g^{ki}(x(t))\xi_k g^{lj}(x(t))\xi_l \\
&= g_{ki}(x(t))\xi_i\xi_k - \frac{1}{2}g^{lk}(x(t))\xi_k\xi_l \\
&= \frac{1}{2}g^{ki}(x(t))\xi_i\xi_k.
\end{aligned}$$

In this way,

$$\begin{aligned}
\frac{\partial H}{\partial \xi_j}(x(t),\xi(t)) &= \frac{1}{2}(g^{ki}(x(t))\delta_i^j\xi_k + g^{ki}(x(t))\xi_i\delta_k^j) \\
&= \frac{1}{2}(g^{kj}(x(t))\xi_k + g^{ji}(x(t))\xi_i) \\
&= g^{ij}(x(t))\xi_i
\end{aligned}$$

and

$$\frac{\partial H}{\partial x^j}(x(t),\xi(t)) = \frac{\partial}{\partial x^j}\left(\frac{1}{2}g^{ki}(x(t))\xi_i\xi_k\right).$$

Therefore $(x(t),\xi(t))$ satisfies

$$\begin{cases} \dot{x}^j(t) = g^{ij}(x(t))\xi_i(t); \\ \dot{\xi}_j(t) = \frac{1}{2}\frac{\partial g^{ik}}{\partial x^j}(x(t))\dot{\xi}_i(t)\dot{\xi}_k(t). \end{cases} \tag{3.4.3}$$

Equation (3.4.3) is the geodesic equation in the Hamiltonian formalism. Observe that it is simpler than the geodesic equation in Lagrangian formalism (3.2.7) due to the absence of a complicated term as the Christoffel symbol.

CHAPTER 4

Hamiltonian Formalism in Riemannian Geometry

In this chapter we apply the Hamiltonian formalism of Section 3.4 to develop the study of curvatures and Jacobi fields in the cotangent space. In Section 4.1 we study the connection on general tensor bundles that comes from the Riemannian connection. This connection will be applied along to the rest of this chapter. In Sections 4.2, 4.3, 4.4 and 4.5 we use the Hamiltonian formalism to give definitions of curvatures for the cotangent bundle and see the equivalence between the definitions given for the tangente bundle. See Chapter 4 and Chapter 7 of [29] for more details of Sections 4.1 and 4.2, respectively.

4.1 Covariant Derivation of Tensor Fields

In this section we define the connection for tensors fields. This concept is the basis to study curvatures and Jacobi fields on the cotangent space.

Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold of dimension n with Riemannian metric $\langle\cdot,\cdot\rangle$. Denote by $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ the Riemannian connection on M. The main idea is to define a connection ∇ on $\mathfrak{X}(M) \times \Gamma(T^{(k,l)}M)$, where $T^{(k,l)}M$ is the fiber bundle of tensors of type (k,l) on M and $\Gamma(T^{(k,l)}M)$ is the space of smooth sections on $T^{(k,l)}M$. This connection has to satisfy the following properties:

1. In $T^{(1,0)}M = TM$, ∇ coincides with the Riemannian connection on M;

2. In $T^{(0,0)}M = M \times \mathbb{R}$, ∇ is given by

$$\nabla_X f = df(X) = Xf;$$

3. ∇ satisfies the product rule with respect to tensor product:

$$\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S).$$

4. ∇ commutes with all contractions, i.e., if $\mathrm{tr}_{(i,j)}$ denote the trace of any pair of index (i, j), then

$$\nabla_X(\mathrm{tr}_{(i,j)} T) = \mathrm{tr}_{(i,j)}(\nabla_X T)$$

This connection satisfies the following additional conditions:

(a) ∇ obeys the product rule:

$$\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y).$$

(b) For any $T \in \Gamma(T^{(k,l)}M)$, smooth 1-forms $\omega^1, \ldots, \omega^k$ and vector fields $Y_1, \ldots, Y_l$ we have that

$$\begin{aligned}(\nabla_X T)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l) =& X(T(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)) \\ &- \sum_{i=1}^{k} T(\omega^1, \ldots, \nabla_X \omega_i, \ldots, \omega^k, Y_1, \ldots, Y_l) \\ &- \sum_{j=1}^{l} T(\omega^1, \ldots, \omega^k, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_l).\end{aligned}$$

First, denote by ∇ a family of connections on $T^{(k,l)}M$ satisfying (1)-(4). We prove that ∇ satisfies (4a) and (4b). Indeed, observe that

$$\begin{aligned}\mathrm{tr}(\omega \otimes Y) &= \mathrm{tr}\left(\omega_i dx^i \otimes Y^j \frac{\partial}{\partial x^j}\right) \\ &= \omega_i Y^j \mathrm{tr}\left(dx^i \otimes \frac{\partial}{\partial x^j}\right) \\ &= \omega_i Y^i \\ &= \omega(Y)\end{aligned}$$

for all $\omega \in \mathfrak{X}^*(M)$ and $Y \in \mathfrak{X}(M)$. Thus, by induction

$$T(\omega^1, \ldots \omega^k, Y_1, \ldots, Y_l) = T^{i_1 \ldots i_k}_{j_1 \ldots j_l} \omega^1_{i_1} \cdots \omega^k_{i_k} Y^{j_1}_1 \cdots Y^{j_l}_l$$

which is represented by

$$\underbrace{\mathrm{tr}\circ\cdots\circ\mathrm{tr}}_{k+l\text{ times}}(T\otimes\omega^1\otimes\cdots\otimes\omega^k\otimes Y_1\otimes\cdots\otimes Y_l)$$

for the sake of simplicity.

$$\begin{aligned}\nabla_X(\omega(Y)) &= \nabla_X(\mathrm{tr}(\omega\otimes Y))\\ &= \mathrm{tr}(\nabla_X(\omega\otimes Y))\\ &= \mathrm{tr}((\nabla_X\omega))\otimes Y+\omega\otimes\nabla_X Y)\\ &= (\nabla_X\omega)(Y)+\omega(\nabla_X Y).\end{aligned}$$

Proceeding by induction we get (4b):

$$\begin{aligned}\nabla_X(T(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l)) &= \nabla_X\mathrm{tr}\circ\cdots\circ\mathrm{tr}(T\otimes\omega^1\otimes\cdots\otimes\omega^k\otimes Y_1\otimes\cdots\otimes Y_l)\\ &= \mathrm{tr}\circ\cdots\circ\mathrm{tr}(\nabla_X(T\otimes\omega^1\otimes\cdots\otimes\omega^k\otimes Y_1\otimes\cdots\otimes Y_l))\\ &= \mathrm{tr}\circ\cdots\circ\mathrm{tr}((\nabla_X T)(\omega^1\otimes\cdots\otimes\omega^k\otimes Y_1\otimes\cdots\otimes Y_l))\\ &+\sum_{i=1}^k\mathrm{tr}\circ\cdots\circ\mathrm{tr}(T\otimes\omega^1\otimes\cdots\otimes\nabla_X\omega^i\otimes\cdots\otimes\omega^k\otimes Y_1\otimes\cdots\otimes Y_l)\\ &+\sum_{j=1}^l\mathrm{tr}\circ\cdots\circ\mathrm{tr}(T\otimes\omega^1\otimes\cdots\otimes\omega^k\otimes Y_1\otimes\cdots\otimes\nabla_X Y_j\otimes\cdots\otimes Y_l).\end{aligned}$$

Since $T(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l)$ is a smooth function, we have that

$$\begin{aligned}(\nabla_X T)(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l) &= X(T(\omega^1,\dots,\omega^k,Y_1,\dots Y_l))\\ &-\sum_{i=1}^k T(\omega^1,\dots,\nabla_X\omega^i,\dots,\omega^k,Y_1,\dots,Y_l)\\ &-\sum_{j=1}^k T(\omega^1,\dots,\omega^k,Y_1,\dots,\nabla_X Y_j,\dots,Y_l).\end{aligned}$$

The next step is prove the uniqueness. By (2) and (4a) the covariant derivative of any 1-form is given by

$$(\nabla_X\omega)(Y)=X(\omega(Y))-\omega(\nabla_X Y),$$

showing that ∇ on $\mathfrak{X}^*(M)$ is only determined by the Riemannian connection on M. Similarly, (4b) gives us an explicit formula of covariant derivative for any $T\in\Gamma(T^{(k,l)}TM)$ in terms of covariant derivatives of smooth vector fields and 1-forms. Therefore, the family of connections ∇ on $T^{(k,l)}M$ is uniquely determined.

For the existence, define the covariant derivative on the 1-forms by

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$$

and for any $T \in \Gamma(T^{(k,l)}M)$ by

$$\begin{aligned}(\nabla_X T)(\omega^1,\ldots,\omega^k,Y_1,\ldots,Y_l) = X(T(\omega^1,\ldots,\omega^k,Y_1,\ldots Y_l))\\ -\sum_{i=1}^{k} T(\omega^1,\ldots,\nabla_X\omega^i,\ldots,\omega^k,Y_1,\ldots,Y_l)\\ -\sum_{j=1}^{k} T(\omega^1,\ldots,\omega^k,Y_1,\ldots,\nabla_X Y_j,\ldots,Y_l).\end{aligned} \tag{4.1.1}$$

We will show that (4.1.1) is multilinear on $\mathscr{D}(M)$ for each ω^i and Y_j, showing that $\nabla_X T$ is a smooth tensor. We prove it for 1-forms and the general case is similar. Given $Y,Z \in \mathfrak{X}(M)$ and $f \in \mathscr{D}(M)$, then

$$\begin{aligned}(\nabla_X \omega)(fY+Z) &= X(\omega(fY+Z)) - \omega(\nabla_X(fY+Z))\\ &= fX(\omega(Y)) + X(\omega(Z)) - \omega(f(\nabla_X Y) + \nabla_X Z)\\ &= f(X(\omega(Y))) - f(\omega(\nabla_X Y)) + X(\omega(Z)) - \omega(\nabla_X Z)\\ &= f((\nabla_X\omega)(Y)) + (\nabla_X\omega)(Z).\end{aligned}$$

and $(\nabla_X \omega)$ is a smooth 1-form because it is linear over $\mathscr{D}(M)$ on $\mathfrak{X}(M)$ and satisfies the product rule on T. By definition of $\nabla_X T$, the proof that $(\nabla_X T)$ is a smooth tensor came as a consequence of the definition of $\nabla_X \omega$ and $\nabla_X Y$, where $\omega \in \mathfrak{X}^*(M)$ and $Y \in \mathfrak{X}(M)$.

Let $T \in \Gamma(T^{(k_1,l_1)}M)$ and $S \in \Gamma(T^{(k_2,l_2)}M)$. By definition of ∇, the property (3) follows by:

$$\begin{aligned}&(\nabla_X(T\otimes S))(\omega^1,\ldots,\omega^{k_1},\ldots,\omega^{k_1+k_2},Y_1,\ldots,Y_{l_1},\ldots,Y_{l_1+l_2}) =\\ &\quad X(T\otimes S(\omega^1,\ldots,\omega^{k_1},\ldots,\omega^{k_1+k_2},Y_1,\ldots,Y_{l_1},\ldots,Y_{l_1+L_2}))\\ &\quad -\sum_{i=1}^{k_1+k_2}(T\otimes S)(\omega^1,\ldots,\nabla_X\omega^i,\ldots,\omega^{k_1+k_2},Y_1,\ldots,Y_{k_1+k_2})\\ &\quad -\sum_{i=1}^{l_1+l_2}(T\otimes S)(\omega^1,\ldots,\omega^{k_1+k_2},Y_1,\ldots,\nabla_X Y_i,\ldots,Y_{l_1+l_2})\end{aligned}$$

$$
\begin{aligned}
&= X(T(\omega^1 \ldots, \omega^{k_1}, Y_1, \ldots, Y_{l_1}) S(\omega^{k_1+1}, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, Y_{l_1+l_2}))\\
&- \sum_{i \le k_1} T(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^{k_1}, Y_1, \ldots, Y_{l_1}) S(\omega^{k_1+1}, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, Y_{l_1+l_2})\\
&- \sum_{i > k_1} T(\omega^1, \ldots, \omega^{k_1}, Y_1, \ldots, Y_{l_1}) S(\omega^{k_1+1}, \ldots, \nabla_X \omega^i, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, Y_{l_1+l_2})\\
&- \sum_{i \le l_1} T(\omega^1, \ldots, \omega^{k_1}, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_{l_1}) S(\omega^{k_1+1}, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, Y_{l_1+l_2})\\
&- \sum_{i > l_1} T(\omega^1, \ldots, \omega^{k_1}, Y_1, \ldots, Y_{l_1}) S(\omega^{k_1+1}, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, \nabla_X Y_i, \ldots, Y_{l_1+l_2})\\
&= X(T(\omega^1, \ldots, \omega^{k_1}, Y_1, \ldots, Y_{l_1})\\
&- \sum_{i \le k_1} T(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^{k_1}, Y_1, \ldots, Y_{l_1})\\
&- \sum_{i \le l_1} T(\omega^1, \ldots, \omega^{k_1}, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_{l_1})) S(\omega^{k_1+1}, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, Y_{l_1+l_2})\\
&+ T(\omega^1, \ldots, \omega^{k_1}, Y_1, \ldots, Y_{l_1})(X(S(\omega^{k_1+1}, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, Y_{l_1+l_2})\\
&- \sum_{i > k_1} S(\omega^{k_1+1}, \ldots, \nabla_X \omega^i, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, Y_{l_1+l_2})\\
&- \sum_{i > l_1} S(\omega^{k_1+1}, \ldots, \omega^{k_1+k_2}, Y_{l_1+1}, \ldots, \nabla_X Y_i, \ldots, Y_{l_1+l_2}))\\
&= ((\nabla_X T) \otimes S + T \otimes (\nabla_X S))(\omega^1, \ldots, \omega^{k_1+k_2}, Y_1, \ldots, Y_{l_1+l_2})),
\end{aligned}
$$

concluding that $\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S)$. Finally, for any $T \in \Gamma(T^{(k,l)}M)$ it follows that

$$
\begin{aligned}
(\nabla_X&(\mathrm{tr}_{(i,j)} T))(\omega^1, \ldots, \omega^{k-1}, Y_1, \ldots, Y_{l_1-1})\\
&= X(\mathrm{tr}_{(i,j)} T(\omega^1, \ldots, \omega^{k-1}, Y_1, \ldots, Y_{l-1}))\\
&- \sum_{m=1}^{k-1} \mathrm{tr}_{(i,j)} T(\omega^1, \ldots, \nabla_X \omega^m, \ldots, \omega^{k-1}, Y_1, \ldots, Y_{l-1})\\
&- \sum_{m=1}^{l-1} \mathrm{tr}_{(i,j)} T(\omega^1, \ldots, \omega^{k-1}, Y_1, \ldots, \nabla_X Y_m, \ldots, Y_{l-1})\\
&= X\left(T\left(\omega^1, \ldots, \omega^{i-1}, dx^s, \omega^i, \ldots, \omega^{k-1}, Y_1, \ldots, Y_{j-1}, \frac{\partial}{\partial x^s}, Y_j, \ldots, Y_{l-1}\right)\right)\\
&- \mathrm{tr}_{(i,j)}\left(\sum_{m=1}^{k-1} T(\omega^1, \ldots, \nabla_X \omega^m, \ldots, \omega^{k-1}, Y_1, \ldots, Y_{l-1})\right)\\
&- \mathrm{tr}_{(i,j)}\left(\sum_{m=1}^{l-1} T(\omega^1, \ldots, \omega^{k-1}, Y_1, \ldots, \nabla_X Y_m, \ldots, Y_{l-1})\right)\\
&= \mathrm{tr}_{(i,j)}(\nabla_X T)(\omega^1, \ldots, \omega^{k-1}, Y_1, \ldots, Y_{l-1}),
\end{aligned}
$$

proving Property (4).

We know that any map $T : \mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathscr{D}(M)$ is a smooth tensor of type (k,l) if and only if is multilinear over $\mathscr{D}(M)$ in $k+l$ entries. Therefore, the map

$$\nabla T : \underbrace{\mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M)}_{k\text{-copies}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l+1\text{-copies}} \to \mathscr{D}(M) \tag{4.1.2}$$

given by

$$(\nabla T)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l, X) = \nabla_X T(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)$$

is a $(k, l+1)$ tensor on M. In fact, we know that $\nabla_X T$ is a smooth tensor, i.e., multilinear over $\mathscr{D}(M)$ in $k+l$ entries. Since ∇ is a connection on $T^{(k,l)}M$, it is linear over $\mathscr{D}(M)$ on X.

Definition 4.1.1. *The tensor (4.1.2) is the **covariant differential of** T.*

Let $\{E_i\}$ be a local frame for M. If ∇ is a Riemannian connection on M we write

$$\nabla_X Y = (X^i Y^j \Gamma^k_{ij} + X(Y^k))E_k, \text{ with } X = X^i E_i, Y = Y^j E_j \in \mathfrak{X}(M),$$

in the local frame $\{E_i\}$ and Γ^k_{ij} are the connection coefficients of ∇ with respect to this frame. The next proposition write the covariant derivative of any tensor with respect to a local frame. Although we use the same notation for Christoffel symbols, this convention will not cause any confusion.

Proposition 4.1.2. *Let M be a Riemannian manifold and ∇ be a family of connections on $T^{(k,l)}M$. Let $\{E_i\}$ be a local frame of M and denote its dual frame by $\{E^i\}$. Let $\{\Gamma^k_{ij}\}$ be the coefficients of ∇ on M with respect to this frame. Let X be a smooth vector field and $X^i E_i$ its expression in local coordinates in terms of this frame.*

1. *The covariant derivative of $\omega = \omega_i E^i \in \mathfrak{X}^*(M)$ is locally given by*

$$\nabla_X \omega = (X(\omega_k) - X^j \omega_i \Gamma^i_{jk})E^k.$$

2. *Let $T \in \Gamma(T^{(k,l)}M)$, whose expression in this local frame is given by*

$$T = T^{i_1 \ldots i_k}_{j_1 \ldots j_l} E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes E^{j_1} \otimes \cdots \otimes E^{j_l}.$$

Then the covariant derivative of T in this local frame is given by

$$\nabla_X T = \left(X\left(T^{i_1\dots i_k}_{j_1\dots j_l}\right) + \sum_{s=1}^{k} X^m T^{i_1\dots p\dots i_k}_{j_1\dots j_l}\Gamma^{i_s}_{mp} - \sum_{s=1}^{l} X^m T^{i_1\dots i_k}_{j_1\dots p\dots j_l}\Gamma^{p}_{mj_s} \right)\cdot$$
$$E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes E^{j_1}\otimes\cdots\otimes E^{j_l}$$

Proof. 1. Let $Y = Y^j E_j \in \mathfrak{X}(M)$. Thus

$$\begin{aligned}(\nabla_X\omega)(Y) &= X(\omega(Y)) - \omega(\nabla_X Y)\\ &= X(\omega_k E^k(Y^j E_j)) - \omega_l E^l(X^j Y^k \Gamma^i_{jk} + X(Y^i))E_i\\ &= X(\omega_k)Y^k + \omega_k(X(Y^k)) - \omega_i(X^j Y^k \Gamma^i_{jk}) - \omega_i X(Y^i))\\ &= (X(\omega_k) - X^j\omega_i\Gamma^i_{jk})Y^k,\end{aligned}$$

implying that $\nabla_X\omega = (X(\omega_k) - X^j\omega_i\Gamma^i_{jk})E^k$.

2. If $T \in \Gamma(T^{(k,l)}M)$, then

$$\begin{aligned}(\nabla_X T)(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l) &= X(T(\omega^1,\dots,\omega^k,Y_1,\dots Y_l))\\ &\quad - \sum_{i=1}^{k} T(\omega^1,\dots,\nabla_X\omega^i,\dots,\omega^k,Y_1,\dots,Y_l)\\ &\quad - \sum_{j=1}^{k} T(\omega^1,\dots,\omega^k,Y_1,\dots,\nabla_X Y_j,\dots,Y_l).\end{aligned}$$

Therefore,

$$\begin{aligned}&(\nabla_X T)(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l)\\ &\quad = X(T^{i_1\dots i_k}_{j_1\dots j_l}E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes E^{j_1}\otimes\cdots\otimes E^{j_l}(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l))\\ &\quad - \sum_{m=1}^{k} T^{i_1\dots i_k}_{j_1\dots j_l}E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes E^{j_1}\otimes\cdots\otimes E^{j_l}(\omega^1,\dots,\nabla_X\omega^m,\dots,\omega^k,Y_1,\dots,Y_l)\\ &\quad - \sum_{n=1}^{l} T^{i_1\dots i_k}_{j_1\dots j_l}E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes E^{j_1}\otimes\cdots\otimes E^{j_l}(\omega^1,\dots,\omega^k,Y_1,\dots,\nabla_X Y_n,\dots,Y_l)\\ &\quad = X(T^{i_1\dots i_k}_{j_1\dots j_l}\omega^1_{i_1}\cdots\omega^k_{i_k}Y^{j_1}_1\cdots Y^{j_l}_l)\\ &\quad - \sum_{m=1}^{k} T^{i_1\dots i_k}_{j_1\dots j_l}E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes E^{j_1}\otimes\cdots\otimes E^{j_l}\\ &\quad (\omega^1,\dots,(X(\omega^m_q) - X^p\omega^m_a\Gamma^a_{pq})E^q,\dots,\omega^k,Y_1,\dots,Y_l)\end{aligned}$$

$$
\begin{aligned}
&-\sum_{n=1}^{l} T^{i_1\ldots i_k}_{j_1\ldots j_l} E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes E^{j_1}\otimes\cdots\otimes E^{j_l}\\
&(\omega^1,\ldots,\omega^k,Y_1\ldots,(X(Y_n^q)+X^pY_n^a\Gamma^q_{pa})E_q,\ldots,Y_l)\\
&=X(T^{i_1\ldots i_k}_{j_1\ldots j_l}\omega^1_{i_1}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots Y_l^{j_l})\\
&-\sum_{m=1}^{k} T^{i_1\ldots i_m\ldots i_k}_{j_1\ldots j_l}\omega^1_{i_1}\cdots(X(\omega^m_{i_m})-X^p\omega^m_a\Gamma^a_{pi_m})\cdots\omega^k_{i_k}Y_1^{j_1}\cdots Y_l^{j_l}\\
&-\sum_{n=1}^{l} T^{i_1\ldots i_k}_{j_1\ldots j_n\ldots j_l}\omega^1_{i_1}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots(X(Y_n^{j_n})+X^pY_n^a\Gamma^{j_n}_{pa})\cdots Y_l^{j_l}\\
&=X(T^{i_1\ldots i_k}_{j_1\ldots j_l})\omega^1_{i_1}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots Y_l^{j_l}\\
&+T^{i_1\ldots i_k}_{j_1\ldots j_l}X(\omega^1_{i_1}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots Y_l^{j_l})\\
&-\sum_{m=1}^{k} T^{i_1\ldots i_k}_{j_1\ldots j_l}\omega^1_{i_1}\cdots X(\omega^m_{i_m})\cdots\omega^k_{i_k}Y_1^{j_1}\cdots Y_l^{j_l}\\
&+\sum_{m=1}^{k} T^{i_1\ldots i_m\ldots i_k}_{j_1\ldots j_l}\omega^1_{i_1}\cdots X^p\omega^m_a\Gamma^a_{pi_m}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots Y_l^{j_l}\\
&-\sum_{n=1}^{l} T^{i_1\ldots i_k}_{j_1\ldots j_n\ldots j_l}\omega^1_{i_1}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots X^pY_n^a\Gamma^{j_n}_{pa}\cdots Y_l^{j_l}\\
&-\sum_{n=1}^{l} T^{i_1\ldots i_k}_{j_1\ldots j_l}\omega^1_{i_1}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots X(Y_n^{j_n})\cdots Y_l^{j_l}.
\end{aligned}
$$

But,

$$
\begin{aligned}
T^{i_1\ldots i_k}_{j_1\ldots j_l}X(\omega^1_{i_1}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots Y_l^{j_l}) &= \sum_{m=1}^{k} T^{i_1\ldots i_k}_{j_1\ldots j_l}\omega^1_{i_1}\cdots X(\omega^m_{i_m})\cdots\omega^k_{i_k}Y_1^{j_1}\cdots Y_l^{j_l}\\
&+\sum_{n=1}^{l} T^{i_1\ldots i_k}_{j_1\ldots j_l}\omega^1_{i_1}\cdots\omega^k_{i_k}Y_1^{j_1}\cdots X(Y_n^{j_n})\cdots Y_l^{j_l}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\nabla_X T = \left(X\left(T^{i_1\ldots i_k}_{j_1\ldots j_l}\right)+\sum_{m=1}^{k} X^h T^{i_1\ldots p\ldots i_k}_{j_1\ldots j_l}\Gamma^{i_m}_{hp}-\sum_{s=1}^{l} X^h T^{i_1\ldots i_k}_{j_1\ldots p\ldots j_l}\Gamma^p_{hj_s}\right)\cdot\\
E_{i_1}\otimes\cdots\otimes E_{i_k}\otimes E^{j_1}\otimes\cdots\otimes E^{j_l}
\end{aligned}
$$

□

To simplify the notation, when we write the components of the covariant differential in terms of a local frame, it is usual to use a semicolon to separate indices resulting from differentiation from the preceding indices. For example, if $Y = Y^i E_i$ is a smooth vector field in the local frame

$\{E_i\}$, then ∇Y is a smooth tensor of type $(1,1)$. Then we write

$$\nabla Y = Y^k_{;i} E_k \otimes E^i.$$

By the expression of the Riemannian connection ∇ in a local frame on M,

$$\begin{aligned}\nabla Y(\omega, X) &= \nabla_X Y(\omega)\\ &= (X^i Y^j \Gamma^k_{ij} + X(Y^k))\omega_k\\ &= (X^i Y^j \Gamma^k_{ij} + X^i E_i(Y^k))\omega_k\\ &= (Y^j \Gamma^k_{ij} + E_i(Y^k)) E_k \otimes E^i(\omega, X).\end{aligned}$$

Since the coefficient is unique, it follows that $Y^i_{;j} = Y^k \Gamma^i_{jk} + E_j(Y^i)$.

In analogous way, if $\omega \in \mathfrak{X}^*(M)$, then $\nabla\omega$ is a tensor of type $(0,2)$. Writing $\nabla\omega = \omega_{i;j} E^i \otimes E^j$, we have that

$$\begin{aligned}\nabla\omega(Y,X) &= (\nabla_X \omega)(Y)\\ &= (X(\omega_k) - X^j \omega_i \Gamma^i_{jk}) Y^k\\ &= (E_j(\omega_k) - \omega_i \Gamma^i_{jk}) E^k \otimes E^j(Y,X),\end{aligned}$$

showing that $\omega_{i;j} = E_j(\omega_i) - \omega_k \Gamma^k_{ji}$.

Applying this convention to the definition of covariant differential for any tensor T of type (k,l), we have that

$$\nabla T = T^{i_1 \dots i_k}_{j_1 \dots j_l; m} E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes E^{j_1} \otimes \cdots \otimes E^m.$$

because ∇T is a tensor of type $(k, l+1)$. By (4b)

$$\begin{aligned}&\nabla T(\omega^1, \dots, \omega^k, Y_1, \dots, Y_l, X)\\ &\quad = (\nabla_X T)(\omega^1, \dots, \omega^k, Y_1, \dots, Y^l)\\ &\quad = \left(X\left(T^{i_1 \dots i_k}_{j_1 \dots j_l}\right) + \sum_{m=1}^{k} X^h T^{i_1 \dots p \dots i_k}_{j_1 \dots j_l} \Gamma^{i_m}_{hp} - \sum_{s=1}^{l} X^h T^{i_1 \dots i_k}_{j_1 \dots p \dots j_l} \Gamma^p_{h j_s} \right) \cdot \omega^1_{i_1} \cdots \omega^k_{i_k} Y^{j_1}_1 \cdots Y^{j_l}_l\\ &\quad = \left(E_h\left(T^{i_1 \dots i_k}_{j_1 \dots j_l}\right) + \sum_{m=1}^{k} T^{i_1 \dots p \dots i_k}_{j_1 \dots j_l} \Gamma^{i_m}_{hp} - \sum_{s=1}^{l} T^{i_1 \dots i_k}_{j_1 \dots p \dots j_l} \Gamma^p_{h j_s} \right) \cdot \omega^1_{i_1} \cdots \omega^k_{i_k} Y^{j_1}_1 \cdots Y^{j_l}_l X^h.\end{aligned}$$

Thus, in the local frame $\{E_i\}$ of M and the convention adopted above we have that

$$T^{i_1\dots i_k}_{j_1\dots j_l;h} = E_h\left(T^{i_1\dots i_k}_{j_1\dots j_l}\right) + \sum_{m=1}^{k} T^{i_1\dots p\dots i_k}_{j_1\dots j_l}\Gamma^{i_m}_{hp} - \sum_{s=1}^{l} T^{i_1\dots i_k}_{j_1\dots p\dots j_l}\Gamma^{p}_{hj_s}.$$

As we define the $(k,l+1)$-tensor ∇T for any (k,l)-tensor T, we can apply again the covariant differential to have a $(k,l+2)$-tensor $\nabla^2 T = \nabla(\nabla T)$, called by **second covariant differential of** T. Given $X,Y \in \mathfrak{X}(M)$, denote by $\nabla^2_{X,Y}T$ the tensor of type (k,l) obtained putting X,Y in the last entries of $\nabla^2 T$, i.e.,

$$(\nabla^2_{X,Y}T)(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l) := \nabla^2 T(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l,Y,X).$$

It is important see that $\nabla^2_{X,Y}T$ is not equal to $\nabla_X(\nabla_Y T)$, because $\nabla^2_{X,Y}T$ is linear over $\mathscr{D}(M)$ on Y, while that $\nabla_X(\nabla_Y T)$ is not.

Proposition 4.1.3. *Let M be a Riemannian manifold and ∇ be a family of connections on $T^{(k,l)}M$. For any $T \in \Gamma(T^{k,l}M)$,*

$$\nabla^2_{X,Y}T = \nabla_X(\nabla_Y T) - \nabla_{\nabla_X Y}T.$$

Proof. The covariant derivative $\nabla_Y T$ can be expressed as the trace of $\nabla T \otimes Y$ in the last two indexes. Indeed, if $\{E_i\}$ is a local frame with dual local frame $\{E^i\}$,

$$\begin{aligned}
\operatorname{tr}(\nabla T\otimes Y)(E^{i_1},\dots,E^{i_k},E_{j_1},\dots,E_{j_l}) &= \nabla T\otimes Y(E^{i_1},\dots,E^{i_k},E^{i},E_{j_1},\dots,E_{j_l},E_i)\\
&= (\nabla T(E^{i_1},\dots,E^{i_k},E_{j_1},\dots,E_{j_l},E_i)Y(E^i))\\
&= T^{i_1\dots i_k}_{j_1\dots j_l;m}Y^m
\end{aligned}$$

By the other side, we just compute that $\nabla_Y T = \sum T^{i_1\dots i_k}_{j_1\dots j_l;m}Y^m$. The general case is a consequence of the multilinearity.

Similarly, $\nabla^2_{X,Y}T = \operatorname{tr}(\operatorname{tr}(\nabla^2 T\otimes X)\otimes Y)$. In fact,

$$\begin{aligned}
&\operatorname{tr}(\operatorname{tr}(\nabla^2 T\otimes X)\otimes Y)(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l)\\
&\qquad= \operatorname{tr}(\nabla^2 T\otimes X)\otimes Y(\omega^1,\dots,\omega^k,E^i,Y_1,\dots,Y_l,E_i)\\
&\qquad= \operatorname{tr}(\nabla^2 T\otimes X)(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l,E_i)Y(E^i)\\
&\qquad= (\nabla^2 T\otimes X)(\omega^1,\dots,\omega^k,E^j,Y_1,\dots,Y_l,E_i,E_j)Y(E^i)
\end{aligned}$$

$$
\begin{aligned}
&= \nabla^2 T(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l,E_i,E_j)X(E^j)Y(E^i) \\
&= \nabla^2 T(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l,E_i,E_j)X^jY^i \\
&= \nabla^2 T(\omega^1,\dots,\omega^k,Y_1,\dots,T_l,Y,X) \\
&= (\nabla^2_{X,Y}T)(\omega^1,\dots,\omega^k,Y_1,\dots,Y_l).
\end{aligned}
$$

Therefore the equality holds. Finally, we have that

$$
\begin{aligned}
\nabla_X(\nabla_Y T) &= \nabla_X(\mathrm{tr}(\nabla T \otimes Y)) \\
&= \mathrm{tr}(\nabla_X(\nabla T \otimes Y)) \\
&= \mathrm{tr}(\nabla_X(\nabla T)\otimes Y + \nabla T \otimes \nabla_X Y) \\
&= \mathrm{tr}(\mathrm{tr}(\nabla\nabla T \otimes X)\otimes Y) + \mathrm{tr}(\nabla T \otimes \nabla_X Y) \\
&= \nabla^2_{X,Y}T + \nabla_{\nabla_X Y}T.
\end{aligned}
$$

□

To make Proposition 4.1.3 more concrete, we have the following example.

Example 4.1.4 (The Hessian)**.** *Let f be a smooth map on M. Then $\nabla f \in \mathfrak{X}^*(M)$. But*

$$\nabla f(X) = \nabla_X f = X(f) = df(X).$$

Therefore, $\nabla f = df$. The $(0,2)$-tensor $\nabla^2 f = \nabla df$ is the ***Hessian of*** *f. We have that*

$$
\begin{aligned}
\nabla^2 f(Y,X) &= \nabla^2_{X,Y} f \\
&= \nabla_X(\nabla_Y f) - \nabla_{\nabla_X Y} f \\
&= \nabla_X(Y(f)) - (\nabla_X Y)(f) \\
&= X(Y(f)) - (\nabla_X Y)f.
\end{aligned}
$$

Let $(x^1,\dots,x^n) : U \to \mathbb{R}^n$ be a coordinate system on an open set $U \subset M$. In local coordinates, $\nabla^2 f = f_{;ij}dx^i \otimes dx^j$, with

$$f_{;ij} = \frac{\partial f}{\partial x^j \partial x^i} - \Gamma^k_{ji}\frac{\partial f}{\partial x^k}.$$

4.2 Curvature Tensor on the Cotangent Bundle

In this section we define the curvature operator on the cotangent bundle and prove relations with the usual definition of the curvature operator on the tangent bundle.

Let $(M, \langle \cdot, \cdot \rangle)$ be an n-dimensional Riemannian manifold. Denote by $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ the curvature operator on M, i.e., the $(1,3)$-tensor defined as

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

The Riezs representation theorem allows us to define the linear map $\flat : TM \to T^*M$ by $v \mapsto \langle v, \cdot \rangle_p$. $\flat$ induces an isomorphism between the tangent and cotangent bundles. The inverse of this isomorphism will be denoted by $\sharp$. These isomorphisms are called **musical isomorphisms**.

For any $X, Y \in \mathfrak{X}(M)$ let $R^*(X,Y) : \mathfrak{X}^*(M) \to \mathfrak{X}^*(M)$ the $(1,1)$-tensor induced by the curvature operator and defined as

$$R^*(X,Y)\xi = (R(X,Y)\xi^\sharp)^\flat.$$

Definition 4.2.1. *The operator $R^*(X,Y)$ will be called by **curvature operator on** T^*M.*

The Riemannian metric on M induces a Riemannian metric in the cotangent bundle by the following identification:

$$\begin{aligned} \langle \cdot, \cdot \rangle^* : T^*M \times T^*M &\to \mathscr{D}(M) \\ (\xi, \eta) &\mapsto \langle \xi^\sharp, \eta^\sharp \rangle. \end{aligned}$$

By definition of $\langle \cdot, \cdot \rangle^*$ we have an isometry between TM and T^*M.

Let $\{E_i\}$ be a local frame on M and denote by $\{E^i\}$ the dual local frame of M. For any $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned} \operatorname{tr}(X \otimes \langle \cdot, \cdot \rangle)(Y) &= (X \otimes \langle \cdot, \cdot \rangle)(E^i, Y, E_i) \\ &= X(E^i)\langle Y, E_i \rangle \\ &= X^i g_{ij} Y^j. \end{aligned}$$

Therefore, $X^\flat = X^i g_{ij} E^j = \mathrm{tr}(X \otimes \langle \cdot, \cdot \rangle)$. Thus,

$$\begin{aligned}\nabla_Y X^\flat &= \nabla_Y \mathrm{tr}(X \otimes \langle \cdot, \cdot \rangle) \\ &= \mathrm{tr}((\nabla_Y X) \otimes \langle \cdot, \cdot \rangle) + \mathrm{tr}(X \otimes \nabla_Y \langle \cdot, \cdot \rangle).\end{aligned}$$

But, $\nabla_Y \langle \cdot, \cdot \rangle (W, Z) = Y(\langle W, Z \rangle) - \langle \nabla_Y W, Z \rangle - \langle W, \nabla_Y Z \rangle = 0$. Then,

$$\begin{aligned}\nabla_Y X^\flat &= \mathrm{tr}((\nabla_Y X) \otimes \langle \cdot, \cdot \rangle) \\ &= (\nabla_Y X)^\flat.\end{aligned} \tag{4.2.1}$$

Putting $X = \xi^\sharp$, we have that

$$(\nabla_Y \xi)^\sharp = \nabla_Y \xi^\sharp$$

because $\flat$ and $\sharp$ are the inverse of each other. The next theorem gives a relation between the second covariant derivative of a 1-form and the curvature operator.

Theorem 4.2.2 (Ricci identity)**.** *Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. The second covariant differential of (k,l)-tensors satisfies the following identities. If $Z \in \mathfrak{X}(M)$, then*

$$\nabla^2_{Y,X} Z - \nabla^2_{X,Y} Z = R(X,Y)Z \tag{4.2.2}$$

If $\beta \in \mathfrak{X}^*(M)$, then

$$\nabla^2_{Y,X} \beta - \nabla^2_{X,Y} \beta = R^*(X,Y)\beta. \tag{4.2.3}$$

Proof. For any $T \in \Gamma(T^{(k,l)}M)$, we have that

$$\begin{aligned}\nabla^2_{Y,X} T - \nabla^2_{X,Y} T &= \nabla_Y \nabla_X T - \nabla_{\nabla_Y X} T - (\nabla_X \nabla_Y T - \nabla_{\nabla_X Y} T) \\ &= \nabla_Y \nabla_X T - \nabla_X \nabla_Y T + \nabla_{[X,Y]} T.\end{aligned}$$

In particular, this equality is true when $T = Z \in \mathfrak{X}(M)$, proving (4.2.2). For $\beta \in \mathfrak{X}^*(M)$ since $\nabla_Y \beta = (\nabla_Y \beta^\sharp)^\flat$, we have that

$$\begin{aligned}\nabla_Y \nabla_X \beta - \nabla_X \nabla_Y \beta + \nabla_{[X,Y]} \beta &= \nabla_Y (\nabla_X \beta^\sharp)^\flat - \nabla_X (\nabla_Y \beta^\sharp)^\flat + (\nabla_{[X,Y]} \beta^\sharp)^\flat \\ &= (\nabla_Y \nabla_X \beta^\sharp)^\flat - (\nabla_X \nabla_Y \beta^\sharp)^\flat + (\nabla_{[X,Y]} \beta^\sharp)^\flat \\ &= (R(X,Y)\beta^\sharp)^\flat \\ &= R^*(X,Y)\beta,\end{aligned}$$

proving (4.2.3). □

The main idea is to use R^* to obtain a curvature tensor on the cotangent bundle. First we need to show that the curvature tensor on T^*M has the linearity in each factor. The operator R^* has the following properties:

1. R^* is $\mathscr{D}(M)$-bilinear on $\mathfrak{X}(M) \times \mathfrak{X}(M)$;
2. For all pair $X, Y \in \mathfrak{X}(M)$, the curvature operator $R^*(X,Y) : \mathfrak{X}^*(M) \to \mathfrak{X}^*(M)$ is linear over $\mathscr{D}(M)$.

Indeed, let $X_1, X_2 \in \mathfrak{X}(M)$ and $f, g \in \mathscr{D}(M)$. It follows by (4.2.3), that

$$\begin{aligned}
R^*(fX_1+gX_2,Y)\xi(Z) &= \nabla^2_{Y,fX_1+gX_2}\xi(Z) - \nabla^2_{fX_1+gX_2,Y}\xi(Z) \\
&= \nabla^2\xi(Z,fX_1+gX_2,Y) - \nabla^2\xi(Z,Y,fX_1+gX_2) \\
&= f\nabla^2\xi(Z,X_1,Y) + g\nabla^2\xi(Z,X_2,Y) - f\nabla^2\xi(Z,Y,X_1) - g\nabla^2(Z,Y,X_2) \\
&= f(\nabla^2_{Y,X_1}\xi(Z) - \nabla^2_{X_1,Y}\xi(Z)) + g(\nabla^2_{Y,X_2}\xi(Z) - \nabla^2_{X_2,Y}\xi(Z)) \\
&= (fR^*(X_1,Y)\xi + gR^*(X_2,Y)\xi)(Z)
\end{aligned}$$

for any $\xi \in \mathfrak{X}^*(M)$ and $Z \in \mathfrak{X}(M)$. In analogous way, we show that

$$R^*(X, fY_1+gY_2)\xi = fR^*(X,Y_1)\xi + gR^*(X,Y_2)\xi.$$

Now, given $\xi, \eta \in \mathfrak{X}^*(M)$ and $f \in \mathscr{D}(M)$, it follows that

$$\begin{aligned}
R^*(X,Y)(\xi+\eta) &= \nabla^2_{Y,X}(\xi+\eta) - \nabla^2_{X,Y}(\xi+\eta) \\
&= (\nabla^2_{Y,X}\xi - \nabla^2_{X,Y}\xi) + (\nabla^2_{Y,X}\eta + \nabla^2_{X,Y}\eta) \\
&= R^*(X,Y)\xi + R^*(X,Y)\eta)
\end{aligned}$$

and

$$R^*(X,Y)(f\xi) = (R(X,Y)(f\xi^\sharp))^\flat = (fR(X,Y)\xi^\sharp)^\flat = fR^*(X,Y)\xi.$$

By the musical isomorphism we can rewrite the dual curvature operator R^* by

$$R^*(\alpha,\beta)\xi := R^*(\alpha^\sharp,\beta^\sharp)\xi \tag{4.2.4}$$

for any $\alpha, \beta, \xi \in \mathfrak{X}^*(M)$.

Proposition 4.2.3 (First Bianchi identity for 1-forms)**.** *For any* $\alpha, \beta, \xi \in \mathfrak{X}^*(M)$*,*

$$R^*(\alpha,\beta)\xi + R^*(\beta,\xi)\alpha + R^*(\xi,\alpha)\beta = 0.$$

Proof. In fact,

$$\begin{aligned} R^*(\alpha,\beta)\xi + R^*(\beta,\xi)\alpha + R^*(\xi,\alpha)\beta &= (R(\alpha^\sharp,\beta^\sharp)\xi^\sharp)^\flat + (R(\beta^\sharp,\xi^\sharp)\alpha^\sharp)^\flat + (R(\xi^\sharp,\alpha^\sharp)\beta^\sharp)^\flat \\ &= 0 \end{aligned}$$

by the First Bianchi identity for vector fields. □

From now on, for any $\alpha, \beta, \xi, \eta \in \mathfrak{X}^*(M)$ we use the following notation:

$$(\alpha,\beta,\xi,\eta)^* := \langle (R^*(\alpha,\beta)\xi)^\sharp, \eta^\sharp \rangle.$$

By the Riemannian metric $\langle \cdot, \cdot \rangle^*$ on the cotangent bundle, we can write

$$(\alpha,\beta,\xi,\eta)^* = \langle R^*(\alpha,\beta)\xi, \eta \rangle^*. \tag{4.2.5}$$

In order to analyze (4.2.5), the action of the musical isomorphism in the covariant derivative of a 1-form, implies that

$$\begin{aligned} (\alpha,\beta,\xi,\eta)^* &= \langle R^*(\alpha,\beta)\xi, \eta \rangle^* \\ &= \langle R(\alpha^\sharp,\beta^\sharp)\xi^\sharp, \eta^\sharp \rangle. \end{aligned} \tag{4.2.6}$$

With this construction, we gain the following proposition.

Proposition 4.2.4. *For any* $\alpha, \beta, \xi, \eta \in \mathfrak{X}^*(M)$ *the follow identities is true:*

1. $(\alpha,\beta,\xi,\eta)^* + (\beta,\xi,\alpha,\eta)^* + (\xi,\alpha,\beta,\eta)^* = 0$;
2. $(\alpha,\beta,\xi,\eta)^* = -(\beta,\alpha,\xi,\eta)^*$;
3. $(\alpha,\beta,\xi,\eta)^* = -(\alpha,\beta,\eta,\xi)^*$;
4. $(\alpha,\beta,\xi,\eta)^* = (\xi,\beta,\alpha,\beta)^*$.

Proof. The proof is a direct consequence of,

$$(\alpha,\beta,\xi,\eta)^* = \langle R(\alpha^\sharp,\beta^\sharp)\xi^\sharp,\eta^\sharp\rangle$$

and (1), (2), (3) and (4) for vector fields $X,Y,Z,W \in \mathfrak{X}(M)$ (see [13]). $\square$

4.3 Sectional Curvature on the Cotangent Bundle

Let $\mathbb{V}$ be a n-dimensional vector space with an inner product $\langle\cdot,\cdot\rangle$. We use the notation $|v\wedge w|$ to indicate the expression

$$\sqrt{|v|^2|w|^2-\langle v,w\rangle^2}. \tag{4.3.1}$$

The expression (4.3.1) represents the area of a bi-dimensional parallelogram generated by $v,w\in\mathbb{V}$. Analogously for all $\xi,\eta\in\mathbb{V}^*$ we define

$$\xi\wedge\eta = (\xi^\sharp\wedge\eta^\sharp)^\flat \tag{4.3.2}$$

and it is straightforward that

$$|\xi\wedge\eta|^* = |(\xi\wedge\eta)^\sharp| = |\xi^\sharp\wedge\eta^\sharp|.$$

Let $(M,\langle\cdot,\cdot\rangle)$ be a Riemannian manifold. Take $p\in M$, for any $\xi,\eta\in T_p^*M$ denote by $\sigma^*\subset T_p^*M$ the bidimensional subspace generated by ξ and η. Consider the number

$$K(\sigma^*) := \frac{(\xi,\eta,\xi,\eta)^*}{(|\xi\wedge\eta|^*)^2},$$

The next proposition is a direct consequence of (4.2.3) and (4.3.2).

Proposition 4.3.1. *Let $\sigma^* = \mathrm{span}\{\xi,\eta\}\subset T_p^*M$ be a bidimensional subspace. Thus, $K(\sigma^*) = K(\sigma)$, where $\sigma = \mathrm{span}\{\xi^\sharp,\eta^\sharp\}$. In particular, $K(\sigma^*)$ doesn't depend on the choice of $\xi,\eta\in\sigma^*$.*

Proof. We can note that

$$K(\sigma^*) = \frac{(\xi,\eta,\xi,\eta)^*}{(|\xi\wedge\eta|^*)^2}$$
$$= \frac{\langle R(\xi^\sharp,\eta^\sharp)\xi^\sharp,\eta^\sharp\rangle}{|\xi^\sharp\wedge\eta^\sharp|^2} = K(\sigma),$$

and the fact that $K(\sigma)$ doesn't depend on the choice of $\xi^\sharp$ and $\eta^\sharp$ (see [13]). □

Definition 4.3.2. *Given $p \in M$ and a bidimensional subspace $\sigma^* \subset T_p^*M$, the real number $K(\xi,\eta) = K(\sigma^*)$ with $\{\xi,\eta\}$ a basis of σ^*, is called **sectional curvature of σ^* in** p.*

The next lemma allows us to characterize when the sectional curvature is constant using the curvature operator on T^*M.

Lemma 4.3.3. *Let $\mathbb{V}$ be a vector space of dimension $n \geq 2$ with an inner product $\langle\cdot,\cdot\rangle$. Let $R : \mathbb{V}^* \times \mathbb{V}^* \times \mathbb{V}^* \to \mathbb{V}^*$ e $R' : \mathbb{V}^* \times \mathbb{V}^* \times \mathbb{V}^* \to \mathbb{V}^*$ trilinear maps such that (1), (2), (3) and (4) of Proposition 4.2.4 holds for*

$$(\alpha,\beta,\xi,\eta) = \langle R^*(\alpha,\beta)\xi,\eta\rangle^*$$

and

$$(\alpha,\beta,\xi,\eta)^{*'} = \langle R^{*'}(\alpha,\beta)\xi,\eta\rangle^*.$$

If α,β are linearly independent, write

$$K(\sigma^*) = \frac{(\alpha,\beta,\alpha,\beta)^*}{(|\alpha\wedge\beta|^*)^2},\ K'(\sigma^*) = \frac{(\alpha,\beta,\alpha,\beta)^{*'}}{(|v\wedge w|^*)^2}$$

with σ^* being the bidimensional subspace generated by α and β. If for every $\sigma^* \subset \mathbb{V}^*$, we have that $K(\sigma^*) = K'(\sigma^*)$, then $R^* = R^{*'}$.

Proof. It is a consequence of Proposition 4.3.1 and the proof of the lemma for vector fields (see [13]). □

As a consequence, we have that:

Proposition 4.3.4. *Let $(M,\langle\cdot,\cdot\rangle)$ be a Riemannian manifold and $p \in M$. Define a trilinear map $R^{*'} : T_p^*M \times T_p^*M \times T_p^*M \to T_p^*M$ by*

$$\langle R^{*'}(\alpha,\beta)\xi,\eta\rangle^* = \langle\alpha,\xi\rangle^*\langle\beta,\eta\rangle^* - \langle\beta,\xi\rangle^*\langle\alpha,\eta\rangle^*.$$

for all $\alpha,\beta,\xi,\eta \in T_p^*M$. Therefore, M has constant sectional curvature equal to K_0 on T^*M if and only if $R^* = K_0R^{*'}$ with R^* being the curvature operator of T^*M.

Proof. For all $\sigma^* \subset T_p^*M$ assume that $K(\sigma^*) = K_0$ and define

$$\langle R^{*'}(\alpha,\beta)\xi,\eta\rangle^* := (\alpha,\beta,\xi,\eta)^{*'}.$$

It is immediately that $R^{*'}$ satisfies (1), (2), (3) and (4) of proposition 4.2.4. As

$$(\xi,\eta,\xi,\eta)^* = \langle\xi,\xi\rangle^*\langle\eta,\eta\rangle^* - (\langle\xi,\eta\rangle^*)^2,$$

for all pair of covectors $\xi,\eta \in T_p^*M$ it follows that

$$\begin{aligned}\langle R^*(\xi,\eta)\xi,\eta\rangle^* &= K_0(\langle\xi,\xi\rangle^*\langle\eta,\eta\rangle^* - (\langle\xi,\eta\rangle^*)^2)\\ &= K_0\langle R^{*'}(\xi,\eta)\xi,\eta\rangle^*.\end{aligned}$$

By the Lemma 4.3.3, for any $\alpha,\beta,\xi,\eta \in T_p^*M$,

$$\langle R^*(\alpha,\beta)\xi,\eta\rangle^* = K_0\langle R^{*'}(\alpha,\beta)\xi,\eta\rangle^*,$$

showing that $R^* = K_0R^{*'}$.

By the other side, let $K^*(\xi,\eta)$ the dual sectional curvature of $\{\xi,\eta\}$ in p with ξ and η linearly independent covectors. Thus,

$$K^*(\xi,\eta) = \frac{(\xi,\eta,\xi,\eta)^*}{(|\xi\wedge\eta|^*)^2} = K_0\frac{(\xi,\eta,\xi,\eta)^{*'}}{(|\xi\wedge\eta|^*)^2} = K_0.$$

□

4.4 Ricci and Scalar Curvatures on Cotangent Bundle

Let $\eta^n = \xi \in T_p^*M$ an unit covector, fix a orthonormal basis $\{\eta^1,\ldots,\eta^{n-1}\}$ of the hyperplane of T_p^*M orthogonal to ξ. Denote by $\{\eta_1,\ldots,\eta_{n-1}\}$ the dual basis of $\{\eta^1,\ldots,\eta^{n-1}\}$

relative to the musical isomorphisms. Define

$$Ric_p^*(\xi) := \frac{1}{n-1}\sum_{i=1}^{n-1}\langle R^*(\xi,\eta_i)\xi,\eta_i\rangle^* = Ric_p(\xi^\sharp) \tag{4.4.1}$$

$$K^*(p) := \frac{1}{n}\sum_{j=1}^{n} Ric_p^*(\eta_j) = \frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i=1}^{n-1}\langle R^*(\eta_j,\eta_i)\eta_j,\eta_i\rangle^* = K(p). \tag{4.4.2}$$

The above expressions is called by **Ricci curvature in the direction of** ξ and **scalar curvature in** p, respectively. Observe that the term scalar curvature at p can be defined independent on the fact that it is calculated on T_pM or on T_p^*M because $K^*(p) = K(p)$. Moreover, by (4.4.1) and (4.4.2), it is clear that $K^*(p)$ and $Ric_p^*(\xi)$ doesn't depend on the choice of the bases.

Define the bilinear form

$$\begin{aligned} Q : T_p^*M \times T_p^*M &\to \mathbb{R} \\ (\xi,\eta) &\mapsto \operatorname{tr}(\omega \mapsto R(\xi^\sharp,\eta^\sharp)^*\omega) \end{aligned}$$

It is well known that the trace does not depend on the choice of coordinates. If ξ is a unitary covector and $\{\eta^1,\dots,\eta^n = \xi\}$ a orthonormal basis of T_p^*M, we have by (4) of 4.2.4 that

$$\begin{aligned} Q(\xi,\eta) &= \sum_{i=1}^{n}(\xi,\xi_i,\eta,\xi_i)^* \\ &= Q(\eta,\xi). \end{aligned}$$

Therefore, Q is a symmetric bilinear form and $Q(\xi,\xi) = (n-1)Ric_p(\xi)$. The tensor Q or $\frac{1}{n-1}Q$ is called by **Ricci tensor on** T^*M.

4.5 Jacobi Fields on Cotangent Bundle

Let $\gamma : [a,b] \to (M,\langle\cdot,\cdot\rangle)$ be a geodesic on M and consider a Jacobi field $J(t)$ along γ, that is, a vector field satisfying

$$\frac{D^2J}{dt^2} + R(\gamma'(t),J(t))\gamma'(t) = 0.$$

Fix a coordinate system on M. On the cotangent bundle, using the Legendre transform, we have that

$$\begin{aligned}
\left(\frac{D^2 J^\flat}{dt^2}\right)_s &= \left(\frac{D^2 J}{dt^2}\right)^\flat_s \\
&= -(R(\gamma'(t), J(t))\gamma'(t))^\flat_s \\
&= -R^i_{jkl}(\gamma')^j(t)J^k(t)(\gamma')^l(t)g_{is} \\
&= -R^i_{jkl}(\gamma')^j(t)J^h(t)g_{hu}g^{uk}(\gamma')^l(t)g_{is} \\
&= -(g_{is}R^i_{jkl}g^{uk})(\gamma')^j(t)(J^\flat)_u(t)(\gamma')^l(t)
\end{aligned}$$

where $J^h(t)g_{hu} = (J^\flat)_u(t)$ and $J^\flat(t)$ is also a solution of a linear ordinary differential equation, which is similar to the Jacobi equation. Therefore these are not any simplification on the Jacobi equation when we consider it on the cotangent bundle.

CHAPTER 5

Euler-Arnold Equations

Let G be a Lie Group, $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. In a Lie group G, it is often possible to represent its invariant geometrical objects on $\mathfrak{g}$ or $\mathfrak{g}^*$. In this Chapter we study a particular case of the geodesic equations on $\mathfrak{g}^*$. Our goal is to extend relations between the bi-invariance of the Haar measure of G and the measure preserving property of the geodesic flow on $\mathfrak{g}^*$. In Section 5.1, we introduce the basic concepts which are needed for our goal. In Section 5.2, we study this problem when the Lie group is equipped with a left invariant Finsler structure. For more details about Haar measure, see [30] and [41]. For unimodular Lie groups, see [31]. For more details about Euler-Arnold equations in Finsler theory, see [2] and [34].

5.1 Unimodular Lie Groups

Let G be an n-dimensional Lie group. Denote by $\mathscr{B}(G)$ the σ-algebra generated by all open subsets of G. This σ-algebra is called **Borel algebra** and its elements is called by **Borel subsets** of G. A measure $\mu : \mathscr{B}(G) \to [0, +\infty]$ is called left invariant if for all Borel subsets $S \in \mathscr{B}(G)$ and all $g \in G$ we have

$$\mu(L_g S) = \mu(S) \tag{5.1.1}$$

for all $g \in G$. The right invariant measure is defined in the same way that (5.1.1) but we apply the right translation in the elements of the Borel algebra.

The existence of a left invariant (right invariant) measure on the Lie group G is guaranteed by the following theorem:

Theorem 5.1.1 (Haar's Theorem)**.** *There exists, up to a positive multiplicative constant, a unique countably additive and nontrivial measure μ on the Borel subsets of G satisfying the* *following conditions:*

1. *The measure μ is left invariant;*

2. *The measure μ is finite on every compact subset $K \subset G$;*

3. *The measure μ is outer regular on Borel subsets $S \subset G$:*

$$\mu(S) = \inf\{\mu(U) : S \subset U \textit{ and } U \textit{ is open in } G\}.$$

4. *The measure μ is inner regular on open subsets $U \subset G$:*

$$\mu(U) = \sup\{\mu(K) : K \subset U \textit{ and } K \textit{ is compact}\}.$$

Proof. See page 165 of [41]. □

A **Haar measure** is a measure μ given in Theorem 5.1.1. If there exists a bi-invariant Haar measure μ, the Lie group G will be called **unimodular**.

In [31], Milnor gives many equivalences for G to be a unimodular Lie group. He states that,

1. The Lie group G is unimodular if and only if the linear transformation $\mathrm{Ad}(g)$ has determinant ± 1 for every $g \in G$, where $\mathrm{Ad}(g) = d(R_{g^{-1}} \circ L_g)$.

2. A connected Lie group G is unimodular if and only if the linear transformation $\mathrm{ad}(v) = [v, \cdot]$ has trace zero for every $v \in \mathfrak{g}$.

Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathfrak{g}$. The Lie algebra structure can be described by an $n \times n \times n$ array of **structures constants** c_{ij}^k where

$$\mathrm{ad}(e_i)(e_j) = c_{ij}^k e_k.$$

If the basis $\{e_1, \ldots, e_n\}$ is orthonormal, then $c_{ij}^k = \langle [e_i, e_j], e_k \rangle$. The second equivalence can be written as

Proposition 5.1.2. *A connected Lie group G with a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ is unimodular if and only if $\mathrm{tr}(\mathrm{ad}(e_i)) = c_{ik}^k$ is zero for all $1 \leq i \leq n$, where $\{e_1, \ldots, e_n\}$ is a basis of $\mathfrak{g}$.*

From now on, we consider G with an auxiliary left invariant Riemannian metric $\langle\cdot,\cdot\rangle$. Let X be a smooth vector field on G with ϕ_t denoting its flow. We say that ϕ_t **preserves the Haar measure μ of G** if

$$\mu(\phi_t(S)) = \mu(S)$$

for every Borel subset S contained in the domain of ϕ_t and for all $t \in I$.

Let $(x^1,\ldots,x^n) : U \to \mathbb{R}^n$ be a local coordinate system on an open set $U \subset G$. A Haar measure restricted to U is a positive scalar multiple of

$$\mu_{\langle\cdot,\cdot\rangle} = \sqrt{\det(g_{ij})}dx^1 \wedge \ldots \wedge dx^n$$

where $g_{ij} = \langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$. The Liouville theorem (see page 21 of [41]) show that the measure $\mu_{\langle\cdot,\cdot\rangle}$ is preserved by the flow ϕ_t of a vector field X on G if and only if $\mathrm{div}X := \mathrm{tr}(Y \mapsto \nabla_X Y)$ is null. It is well known that the divergence of $X = X^i\frac{\partial}{\partial x^i}$ is given by

$$\mathrm{div}(X) = \frac{1}{\sqrt{\det(g_{ij})}}\frac{\partial}{\partial x^k}\left(X^k\sqrt{\det(g_{ij})}\right)$$

and depends only on the volume element induced by the Riemannian metric (see [29]). Therefore the results in this chapter about flows that preserves the volume element will not depend on the choice of $\langle\cdot,\cdot\rangle$.

5.2 Euler-Arnold Equations in Finsler Theory

In 1966, Vladimir Arnold observed in [4] that many basic equations in physics can be seen as equations in Lie groups with a left invariant Riemannian metric. The aim of Arnold was the study of equations in hydrodynamics. For this, Arnold describe the geodesics equations using the Hamiltonian formalism and the left invariant metric as

$$\alpha'(t) = \alpha(t)([\alpha(t)^\sharp, \cdot]) \tag{5.2.1}$$

with $\alpha : I \to \mathfrak{g}^*$ is a smooth curve on the dual Lie algebra $\mathfrak{g}^*$. The first order differential equation (5.2.1) is called **Euler-Arnold equation**.

In [25], Kozlov showed that the solution of the Euler-Arnold equations preserve the Haar measure on $\mathfrak{g}^*$ if and only if the Lie group G is unimodular Lie group. In terms of hydrodynamics, Koslov showed that the flow of Euler-Arnold equation is an incompressible fluid if and only if G is a unimodular group.

The aim of this section is to generalize the results obtained by Kozlov for Finsler theory. Let $F : \mathfrak{g} \to \mathbb{R}$ be a left invariant Minkowski norm on G. Given $v \in \mathfrak{g} \setminus \{0\}$, define the element $v^\flat$ by

$$v^\flat(w) := g_{ij}(v)v^i w^j$$

where g_{ij} is the Hessian matrix of F^2. In page 407 of [6] we have the following proposition.

Proposition 5.2.1. *Let F_* denote the dual norm on $\mathfrak{g}^*$, as defined in (3.3.5). Then:*

1. *The Legendre transform $v \mapsto v^\flat$ if a smooth diffeomorphism from $\mathfrak{g} \setminus \{0\}$ onto $\mathfrak{g}^* \setminus \{0\}$;*

2. *It is a norm preserving. That is,*
$$F_*(v^\flat) = F(v);$$

3. *The inverse of the Legendre transform is given by*
$$\xi_i \mapsto \xi^i := g^{ij}(\xi)\xi_j;$$

4. *At $\xi = v^\flat$, we have*
$$g^{ij}(\xi) = g^{ij}(v)$$
where
$$g^{ij}(\xi) := \left[\frac{1}{2}F_*^2(\xi)\right]_{\xi_i \xi_j} \tag{5.2.2}$$
and $g^{ij}(v)$ denotes the inverse matrix of $g_{ij}(v)$;

Since the Legendre transform on a Finsler manifold (M,F) depends only on F restricted to each tangent space, Proposition 5.2.1 can be naturally adapted to a Finsler manifold. For a generalization of this setting for some C^0-Finsler manifolds see [37].

From now on, consider a Lie group endowed with a left invariant Finsler structure F.

Let $\langle \cdot, \cdot \rangle$ be an auxiliary left invariant Riemannian metric on G. We can see the Minkowski sphere S_{F_*} as a submanifold of the Lie algebra $\mathfrak{g}^*$ of codimension 1 given by the inclusion $i : S_{F_*} \hookrightarrow \mathfrak{g}^*$.

Let $\{e_1, \ldots, e_{n+1}\}$ be a basis of the Lie algebra $\mathfrak{g}$. Denote by $\{e^1, \ldots, e^{n+1}\}$ the dual basis of $\mathfrak{g}^*$. In [2] and [34] the Euler-Arnold equation on Lie groups endowed with C^0-Finsler structures is calculated. In addition, it is shown that the solutions of the Euler-Arnold equation are tangent to the sphere S_{F^*} because the Hamiltonian H is the energy map on $\mathfrak{g}^*$ and the energy is constant along the solutions of Hamilton equations. The Euler-Arnold equation on $\mathfrak{g}^*$ when F is a Finsler structure is given by

$$\begin{aligned} \mathscr{E} : \mathfrak{g}^* &\to \mathfrak{g}^* \\ \alpha &\mapsto \alpha([\alpha^\sharp, \cdot]) \end{aligned} \tag{5.2.3}$$

Denote by $\mathscr{E}_{S_{F_*}} = \mathscr{E}|_{S_{F_*}}$. If we write $\alpha = \alpha_i e^i$, in local coordinates the vector field (5.2.3) is given by

$$\begin{aligned} \mathscr{E}(\alpha)e_k &= \alpha([\alpha^\sharp, e_k]) \\ &= \alpha_j e^j([\alpha^i e_i, e_k]) \\ &= \alpha_j \alpha^i c^j_{ik}. \end{aligned} \tag{5.2.4}$$

Let $\overline{\nabla}$ be a Riemannian (flat) connection on $\mathfrak{g}^*$ with respect to the auxiliary inner product $\langle \cdot, \cdot \rangle^*$, denote by ∇ the induced Riemannian connection on S_{F_*} (see Section 1.4). Given $\alpha \in S_{F_*}$ denote by $\{E^1, \dots, E^{n+1}\}$ the orthonormal frame of $\mathfrak{g}^*$ in a neighborhood U of α such that $\{E^1, \dots, E^n\}$ is a orthonormal frame of S_{F_*}. The shape operator of S_{F_*} in $\mathfrak{g}^*$ with respect to $E^{n+1}(\alpha)$ is given by:

$$\begin{aligned} A_{E^{n+1}(\alpha)} : T_\alpha S_{F_*} &\to T_\alpha S_{F^*} \\ \eta &\mapsto -\left(\overline{\nabla}_\eta E^{n+1}(\alpha)\right)^T. \end{aligned}$$

The shape operator is a symmetric operator (see [13]). Thus there exist a orthonormal basis of eigenvectors $\{e^1, \dots, e^n\}$ of $T_\alpha S_{F_*}$ and principal curvatures $k_1, \dots, k_n$. The vector field E^{n+1} of $(TS_{F^*})^\perp$ can be decomposed as

$$E^{n+1}(\alpha) = \varphi\alpha + Y^{n+1}(\alpha) \tag{5.2.5}$$

where $\varphi \in \mathscr{D}(G)$ and $Y^{n+1}(\alpha) \in T_\alpha S_{F_*}$. Write $Y^{n+1}(\alpha) = y_l E^l(\alpha)$. Extending the vector field E^{n+1} radially we have that

$$(\overline{\nabla}_\alpha E^{n+1})(\alpha) = 0. \tag{5.2.6}$$

Thus,

$$
\begin{aligned}
(\mathrm{div}_{\mathfrak{g}^*}\mathscr{E})(\alpha) &= \sum_{l=1}^{n+1} \langle \overline{\nabla}_{E^l}\mathscr{E}, E^l\rangle^*(\alpha) \\
&= \sum_{l=1}^{n} \langle \overline{\nabla}_{E^l}\mathscr{E}, E^l\rangle^*(\alpha) + \langle \overline{\nabla}_{E^{n+1}}\mathscr{E}, E^{n+1}\rangle^*(\alpha) \\
&= \sum_{l=1}^{n} \langle \left(\overline{\nabla}_{E^l}\mathscr{E}\right)^T + \left(\overline{\nabla}_{E^l}\mathscr{E}\right)^\perp, E^l\rangle^*(\alpha) + \langle \overline{\nabla}_{E^{n+1}}\mathscr{E}, E^{n+1}\rangle^*(\alpha) \\
&= \sum_{l=1}^{n} i^*\langle \nabla_{E^l}\mathscr{E}, E^l\rangle^*(\alpha) + \langle \overline{\nabla}_{E^{n+1}}\mathscr{E}, E^{n+1}\rangle^*(\alpha) \\
&= \mathrm{div}\mathscr{E}_{S_{F_*}}(\alpha) + \langle \overline{\nabla}_{E^{n+1}}\mathscr{E}, E^{n+1}\rangle^*(\alpha) \\
&= \mathrm{div}\mathscr{E}_{S_{F_*}}(\alpha) - \langle \mathscr{E}, \overline{\nabla}_{E^{n+1}}E^{n+1}\rangle^*(\alpha) \\
&= \mathrm{div}\mathscr{E}_{S_{F_*}}(\alpha) + \langle \mathscr{E}(\alpha), S_{E^{n+1}(\alpha)}(Y^{n+1}(\alpha))\rangle^* - \langle \mathscr{E}(\alpha), \varphi\overline{\nabla}_\alpha E^{n+1}\rangle^* \\
&= \mathrm{div}\mathscr{E}_{S_F^*}(\alpha) + \langle \alpha_j\alpha^i c_{im}^j E^m(\alpha), y_l A_{E^{n+1}(\alpha)}(E^l(\alpha))\rangle
\end{aligned}
$$

due to (5.2.4), (5.2.5) and (5.2.6). It follows that

$$
\begin{aligned}
(\mathrm{div}_{\mathfrak{g}^*}\mathscr{E})(\alpha) &= \mathrm{div}\mathscr{E}_{S_F^*}(\alpha) + \alpha_j\alpha^i c_{im}^j y_l \langle E^m(\alpha), k_l E^l(\alpha)\rangle \\
&= \mathrm{div}\mathscr{E}_{S_{F_*}}(\alpha) + \sum_{l=1}^{n} \alpha_j\alpha^i c_{im}^j y_l k_l \delta^{ml} \\
&= \mathrm{div}\mathscr{E}_{S_{F_*}}(\alpha) + \sum_{m=1}^{n} (\alpha_j\alpha^i c_{im}^j y_m k_m)(\alpha). \qquad (5.2.7)
\end{aligned}
$$

Equation (5.2.7) allow us to extend the result proved by Kozlov as:

Theorem 5.2.2. *The Euler-Arnold equations in the Riemannian sphere $S_{\langle\cdot,\cdot\rangle^*}$ preserve the Haar measure if and only if G is unimodular.*

Proof. If the Minkowski norm F is the left invariant Riemannian metric $\langle\cdot,\cdot\rangle$, the vector field (5.2.5) is $E^{n+1}(\alpha) = \alpha$. Thus, the equation 5.2.7 is given by

$$\mathrm{div}\mathscr{E}_{S_{F_*}}(\alpha) = (\mathrm{div}_{\mathfrak{g}^*}\mathscr{E})(\alpha).$$

By [25], $(\mathrm{div}_{\mathfrak{g}^*}\mathscr{E})(\alpha) = 0$ if and only if G is unimodular and the result follows. □

Equation (5.2.7) states that if the vector field E^{n+1} is in the radial direction, that is, if for all $\alpha \in S_{F^*}$ the normal vector field is given by $E^{n+1}(\alpha) = \varphi\alpha$ for some $\varphi \in \mathscr{D}(M)$, then $\mathrm{div}_{\mathfrak{g}^*}\mathscr{E} = \mathrm{div}\mathscr{E}_{S_F^*}$.

Consider the map

$$\begin{aligned} \mathrm{dist}_0 : S_{F_*} &\to \mathbb{R} \\ \alpha &\mapsto |\alpha|^* \end{aligned}$$

where $|\alpha|^* = \sqrt{\langle \alpha, \alpha \rangle^*}$. Since S_{F_*} is a compact set of $\mathfrak{g}^*$, the map dist_0 has a critical point. Let α_0 be a critical point of dist_0 on S_{F_*}. Then, $E^{n+1}(\alpha_0) = \varphi\alpha_0$ for some $\varphi \in \mathscr{D}(M)$. Indeed, let $v \in T_{\alpha_0}S_{F_*}$. Then there exists a smooth curve $\gamma : I \to S_{F_*}$ such that $\gamma(0) = \alpha_0$ and $\gamma'(0) = v$. Thus,

$$\begin{aligned} 0 &= \frac{1}{2}(d(\mathrm{dist}_0^2)_{\alpha_0})(v) \\ &= \frac{1}{2}\left(\frac{d}{dt}\mathrm{dist}_0^2(\alpha(t))\Big|_{t=0}\right) \\ &= \frac{1}{2}\left(\frac{d}{dt}\langle \gamma(t), \gamma(t) \rangle\Big|_{t=0}\right) \\ &= \langle v, \alpha_0 \rangle, \end{aligned}$$

showing that α_0 is normal to $T_{\alpha_0}S_{F_*}$. This implies the following

Proposition 5.2.3. *If $\alpha_0 \in S_{F_*}$ is a critical point of the map* dist_0*, then*

$$\mathrm{div}\mathscr{E}_{S_{F_*}}(\alpha_0) = \mathrm{div}_{\mathfrak{g}^*}\mathscr{E}(\alpha_0).$$

As a direct consequence, we have

Corollary 5.2.4. *If the Finsler structure F_* on $\mathfrak{g}^*$ is Riemannian, then*

$$\mathrm{div}\mathscr{E}_{S_{F_*}} = \mathrm{div}_{\mathfrak{g}^*}\mathscr{E}.$$

Proof. It is enough to consider $\langle \cdot, \cdot \rangle^*$ as the inner product correspondent to F_*. □

We can analyze the divergence of the Euler-Arnold equations considering an arbitrary basis $\{e^1, \ldots, e^{n+1}\}$ of $\mathfrak{g}^*$ and an auxiliary inner product $\langle \cdot, \cdot \rangle^*$ on $\mathfrak{g}^*$ such that $\{e^1, \ldots, e^{n+1}\}$ is an orthonormal basis. In local coordinates, the vector field $\mathscr{E}$ defined on $\mathfrak{g}^*$ with respect to a

Minkowski norm F is:

$$\begin{aligned}\mathscr{E}(\alpha)e_k &= \alpha([\alpha^\sharp, e_k])\\ &= \alpha_j e^j([\alpha_s g^{sl} e_l, e_k])\\ &= \alpha_j \alpha_s g^{sl} c^j_{lk},\end{aligned}$$

where g^{is} is the inverse of the fundamental tensor of F. If $\overline{\nabla}$ is Riemannian flat connection on $\mathfrak{g}^* \setminus \{0\}$ with respect to the metric $\langle \cdot, \cdot \rangle^*$, we have that $\langle \overline{\nabla}_{e^l} e^k, e^l \rangle = 0$ and

$$\begin{aligned}(\mathrm{div}_{\mathfrak{g}\setminus\{0\}}\mathscr{E})(\alpha) &= \langle \overline{\nabla}_{e^l} \alpha_j \alpha_s g^{is} c^j_{ik} e^k, e^l \rangle^*\\ &= e^l(\alpha_j \alpha_s g^{is} c^j_{ik}) \langle e^k, e^l \rangle^* + \alpha_j \alpha_s g^{is} c^j_{ik} \langle \overline{\nabla}_{e^l} e^k, e^l \rangle^*\\ &= \alpha_s g^{is} c^j_{ik} e^k(\alpha_j) + \alpha_j g^{is} c^j_{ik} e^k(\alpha_s) + \alpha_j \alpha_s c^j_{ik} e^k(g^{is})\\ &= \alpha_s g^{is} c^k_{ik} + \alpha_j g^{is} c^j_{is} + \alpha_j \alpha_s c^j_{ik} C^{kis}\end{aligned}$$

By the Schwartz theorem and (5.2.2) of Proposition 5.2.1, C^{kis} is symmetric with respect to k, i and s. Moreover, g^{is} is symmetric and c^j_{is} is anti-symmetric with respect to i and s. Thus $g^{is} c^j_{is}$ and $c^j_{ik} C^{kis}$ are always zero. Then,

$$(\mathrm{div}_{\mathfrak{g}\setminus\{0\}}\mathscr{E})(\alpha) = a_s g^{is} c^k_{ik}. \tag{5.2.8}$$

If G is a unimodular group, then $c^k_{lk} = 0$ for every $l = 1, \ldots, n$ and $\mathrm{div}_{\mathfrak{g}\setminus\{0\}}(\mathscr{E}) \equiv 0$. Reciprocally, if $c^k_{sk} \neq 0$ for some $s = 1, \ldots, k$, then consider α_s proportional to c^k_{sk} and observe that $(\mathrm{div}_{\mathfrak{g}\setminus\{0\}}(\mathscr{E}))(\alpha) = \alpha_s g^{is} c^k_{ik} > 0$. Therefore we proved the following result.

Theorem 5.2.5. *Let G be a Lie group endowed with a left invariant Finsler structure. Then the flow correspondent to the Euler-Arnold equations on $\mathfrak{g}^*$ preserves the Haar measure on $\mathfrak{g}^* \setminus \{0\}$ if and only if G is unimodular.*

Final Considerations

In this chapter we write a little about what was done in this work and what can be done in the future.

The Chapter 4 tells us that geometrical invariants such as curvatures and Jacobi fields in Riemannian manifolds on the cotangent bundle do not have a simpler formula compared with their version on tangent bundle. These facts are not helpful, in contrast to the geodesic equation on the cotangent bundle which is simpler than on TM. It would be interesting to find an example and eventually put this equation in a computer to see if the computer needs less computational power to find or estimate geodesics.

Chapter 5 gives us Equation (5.2.7). We have some observations to do about the term

$$\sum_{m=1}^{n} \alpha_j \alpha^i c_{im}^j y_m k_m. \tag{5.2.9}$$

This term depends on the extension of the orthonormal frame in the radial direction and of the principal curvatures of the Finsler sphere. Corollary 5.2.4 states that if the Finsler structure F_* is Riemannian then $\mathrm{div}\mathscr{E}_{S_{F_*}} = \mathrm{div}_{\mathfrak{g}^*}\mathscr{E}$. Therefore it makes sense to ask if the converse holds: If the divergence on $\mathfrak{g}^*$ coincides with the divergence on S_{F_*}, then the left invariant Finsler structure on the Lie group it Riemannian.

In the same direction, it can be discussed what means the term (5.2.9). Proposition 5.2.3 states that if a point of the sphere S_{F^*} is a critical point of the distance map dist_0, then (5.2.9) is zero. It makes sense trying to see (5.2.9) as a geometric object related to the geometry of $S_{F_*} \subset \mathfrak{g}^*$.

Index

Bibliography

[1] Ralph Abraham and Jerrold E. Marsden. *Foundations of mechanics*. American Mathematical Soc., 2008.

[2] Andrei A. Agrachev and Yuri Sachkov. *Control theory from the geometric viewpoint*, volume 87. Springer Science & Business Media, 2013.

[3] Peter L. Antonelli, Roman S. Ingarden, and Makoto Matsumoto. *The theory of sprays and Finsler spaces with applications in physics and biology*, volume 58. Springer Science & Business Media, 1993.

[4] Vladimir I. Arnold. Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Annales de l'institut Fourier*, 16(1):319–361, 1966.

[5] Vladimir I. Arnold. *Vladimir I. Arnold-Collected Works: Hydrodynamics, Bifurcation Theory, and Algebraic Geometry 1965-1972*. Springer Science & Business Media, 2013.

[6] David Bao, S-S Chern, and Zhongmin Shen. *An introduction to Riemann-Finsler geometry*, volume 200. Springer Science & Business Media, 2012.

[7] Haim Brézis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, 2011.

[8] Élie Cartan. *Les espaces de Finsler*. Number N° 79 in Actualités scientifiques et industrielles. Hermann, 1934.

[9] Stefan Cobzas. *Functional analysis in asymmetric normed spaces*. Springer Science & Business Media, 2012.

[10] Ana Cannas Da Silva. *Lectures on symplectic geometry*, volume 3575. Springer, 2008.

[11] Marcos Dajczer and Ruy Tojeiro. *Submanifold Theory*. Springer, 2019.

[12] Manfredo P. do Carmo. *Differential geometry of curves and surfaces: revised and updated second edition*. Courier Dover Publications, 2016.

[13] Manfredo P. do Carmo and J. Flaherty Francis. *Riemannian geometry*, volume 6. Springer, 1992.

[14] Ehtibar N. Dzhafarov and Hans Colonius. Multidimensional fechnerian scaling: Basics. *Journal of Mathematical Psychology*, 45(5):670–719, 2001.

[15] Leonhard Euler. *Theoria motus corporum solidorum seu rigidorum*. AE Roser., 1765.

[16] Paul Finsler. *Über kurven und flächen in allgemeinen räumen*, volume 11. Springer-Verlag, 2013.

[17] Ryuichi Fukuoka and Anderson M. Setti. Mollifier smoothing of C^0-finsler structures. *Annali di Matematica Pura ed Applicata (1923-)*, 200(2):595–639, 2021.

[18] Carl Friedrich Gauss. *Disquisitiones generales circa superficies curvas*, volume 1. Typis Dieterichianis, 1828.

[19] Branko Grünbaum. *Convex Polytopes*, volume 221. Springer Science & Business Media, 2003.

[20] William R. Hamilton. *On a general method in dynamics*. Richard Taylor, 1834.

[21] Godfrey H. Hardy, John E. Littlewood, and George Pólya. *Inequalities*. Cambridge university press, 1952.

[22] Allen Hatcher. *Algebraic topology*. Cambridge Univ. Press, Cambridge, 2000.

[23] Roman Ingarden. On physical applications of Finsler geometry. *Contemporary Mathematics*, 196:213–226, 1996.

[24] Jürgen Jost and Xianqing Li-Jost. *Calculus of variations*, volume 64. Cambridge University Press, 1998.

[25] Valery V. Kozlov. Invariant measures of Euler–Poincaré equations on Lie algebras. *Funktsional'nyi Analiz i ego Prilozheniya*, 22(1):69–70, 1988.

[26] László Kozma and Lajos Tamássy. Finsler geometry without line elements faced to applications. *Reports on Mathematical Physics*, 51(2-3):233–250, 2003.

[27] Joseph L. Lagrange. *Mécanique analytique*, volume 1. Mallet-Bachelier, 1853.

[28] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[29] John M Lee. *Introduction to Riemannian manifolds*. Springer, 2018.

[30] Ricardo Mané. *Ergodic theory and differentiable dynamics*, volume 8. Springer Science & Business Media, 2012.

[31] John Milnor. Curvatures of left invariant metrics on lie groups. *Advances in mathematics*, 21(3):293–329, 1976.

[32] Lev S. Pontryagin. The mathematical theory of optimal processes and differential games. *Trudy Matematicheskogo Instituta imeni VA Steklova*, 169:119–158, 1985.

[33] Jéssica B. Prudencio. *Extremals on three-dimensional Lie groups endowed with left invariant polyhedral Finsler structures*. PhD thesis, Universidade Estadual de Maringá, 2022.

[34] Jéssica B. Prudencio and Ryuichi Fukuoka. Extremals on three-dimensional lie groups with asymmetric polyhedral finsler structures. *arXiv preprint arXiv:2204.02465*, 2022.

[35] Bernhard Riemann. Über die hypothesen, welche der geometrie zu grunde liegen. *Physikalische Blätter*, 10(7):296–306, 1954.

[36] Hugo M. Rodrigues. *Geodesic fields for Pontryagin type C^0-Finsler manifolds*. PhD thesis, Universidade Estadual de Maringá, 2020.

[37] Hugo M. Rodrigues and Ryuichi Fukuoka. Geodesic fields for Pontryagin type C^0-Finsler manifolds. *ESAIM: Control, Optimisation and Calculus of Variations*, 28:19, 2022.

[38] Hanno Rund. *The differential geometry of Finsler spaces*, volume 101. Springer Science & Business Media, 2012.

[39] Michael Spivak. *A comprehensive introduction to differential geometry, vol. I.* Publish or Perish, 3rd edition, 1999.

[40] Elias M. Stein and Rami Shakarchi. *Fourier analysis: an introduction*, volume 1. Princeton University Press, 2011.

[41] Marcelo Viana and Krerley Oliveira. *Foundations of Ergodic Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.